Nakajima quiver varieties

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Contents

1 Introduction

These notes are written for a pre-talk for the Harvard-MIT Algebraic Geometry Seminar. The goal of these notes is to discuss Nakajima's quiver varieties. However, I absorbed the notes I wrote for a previous talk on stability conditions for quiver representations as they were relevant.

These notes are heavily based on [KJ16], especially for the first two sections. Also very useful (and very similar exposition) is [Gin09]; of course, there's also notes from the man himself, such as [Nak16].

1.1 Conventions

We always work over the ground field $\mathbb{C}$.

2 Quiver representations

2.1 Quivers

First, a brief overview of generalities on quivers. For more details, see [KJ16].

Definition 2.1 (quiver): A **quiver** is a directed graph.

We typically denote a quiver by $\overrightarrow{Q}$. We'll denote the set of vertices by I and the set of directed edges by Ω. For an edge $e \in \Omega$, we'll write $e : i \to j$ to indicate that it goes from i to j, and write e_h to denote the head of e, and e_t to denote the tail of e. If we forget the directions of the edges in $\overrightarrow{Q}$, then we get a graph, which we will denote by Q.

For our purposes, we always assume that I, Ω are **finite**, and that $\overrightarrow{Q}$ is **connected**.

Definition 2.2 (acyclic): We say a quiver $\overrightarrow{Q}$ is **acyclic** if it has no oriented cycles.

2.2 Representations of quivers

Definition 2.3 (representation): Let $\overrightarrow{Q}$ be a quiver. Then a **representation** V of $\overrightarrow{Q}$ is the data of:

- Vector spaces V_i for each $i \in I$,
- Linear maps $x_e : V_i \to V_j$ for each $e : i \to j \in \Omega$.

We only consider **finite-dimensional representations** in these notes.

Definition 2.4: A **morphism of representations** $f : V \to W$ is a collection of operators $f_i : V_i \to W_i$ for each $i \in I$, which commute with the operators x_e. The morphisms $V \to W$ form a vector space and we denote it by $\mathrm{Hom}_{\overrightarrow{Q}}(V, W)$, or simply by $\mathrm{Hom}(V, W)$.

Definition 2.5 (Rep $\overrightarrow{Q}$): These two combine to give us the **category of (finite-dimensional) representations** of a quiver $\overrightarrow{Q}$, which we denote by Rep $\overrightarrow{Q}$. We denote its bounded derived category by $D^b(\overrightarrow{Q}) \coloneqq D^b(\text{Rep}\,\overrightarrow{Q})$.

Rep $\overrightarrow{Q}$ has many of the standard operations that we're familiar with; these include direct sums, subrepresentations, quotient representations, kernels, and images. This makes Rep $\overrightarrow{Q}$ into an $\mathbb{C}$**-linear abelian category**. Actually, the reason for this is due to the path algebra.

2.3 Path algebra

Fix a quiver $\overrightarrow{Q}$. A **path** in $\overrightarrow{Q}$ is just a sequence of edges so that the tail of one edge is the head of the next. The **length** of a path is just the number of edges in the sequence. We allow for length 0 paths, which we denote by e_i for $i \in I$. Finally, we define **multiplication of paths** by concatenating them if the tail of first path is the head of the second path, and zero otherwise.

Definition 2.6 (path algebra): The **path algebra** $\mathbb{C}\overrightarrow{Q}$ is the $\mathbb{C}$-algebra with basis given by all paths in $\overrightarrow{Q}$ (including length 0), with multiplication given by multiplication of paths.

Here are some immediate properties:

a) $\mathbb{C}\overrightarrow{Q}$ is an associative algebra with unit $1 = \sum_{i\in I} e_i$.

b) $\mathbb{C}\overrightarrow{Q}$ is naturally $\mathbb{Z}_{\geq 0}$-graded by path length.

c) $\mathbb{C}\overrightarrow{Q}$ is finite-dimensional iff $\overrightarrow{Q}$ contains no oriented cycles.

d) The length-zero paths e_i are indecomposable projections summing to 1.

The important point is that:

Theorem 2.7: The category of $\overrightarrow{Q}$-representations, not necessarily finite-dimensional, is equivalent to the category of left $\mathbb{C}\overrightarrow{Q}$-modules.

If we start with a $\overrightarrow{Q}$-representation V, then we get a $\mathbb{C}\overrightarrow{Q}$-module M by setting $M \coloneqq \bigoplus_{i\in I} V_i$, with the path $e \in \Omega$ acting by x_e. On the other hand, from a $\mathbb{C}\overrightarrow{Q}$-module M, we recover a $\overrightarrow{Q}$-representation V by setting $V_i \coloneqq e_i M$, and for $e : i \longrightarrow j \in \Omega$, setting x_e to be the operator induced by the path e, sending $e_i M \longrightarrow e_j M$ since $e = e_j \cdot e \cdot e_i \in \mathbb{C}\overrightarrow{Q}$.

Therefore, we easily see that Rep $\overrightarrow{Q}$ is abelian and inherits all the usual notions from the theory of modules over associative algebras. In particular:

Definition 2.8: We have notions of **simple**, **semisimple**, and **indecomposable** representations in Rep $\overrightarrow{Q}$. Write Ind$(\overrightarrow{Q})$ to be the set of isomorphism classes of nonzero indecomposable representations of Rep $\overrightarrow{Q}$.

Theorem 2.9: The simple representations of an acyclic quiver are $S(i)$ for $i \in I$, which are the representations given by a single one-dimensional vector space at vertex i, zero for every other vertex, and every edge is the zero operator.

Proof. Suppose V is a simple representation. Pick some vertex $i \in I$ such that $V_i \neq 0$, and i is "maximal" in the sense that for every edge $i \longrightarrow j$, then $V_j = 0$. This can be done because there are no oriented cycles. Then V_i itself is a subrepresentation (change every other vector space to 0 and every edge to the zero operator; we also are abusing notation here). By simplicity of V, it must be true that $V = V_i$.
On the other hand, it's obvious that every $S(i)$ is simple. ☐

So the simple representations are easy to describe, which is good because of theorems like Jordan-Hölder. However, the indecomposable representations are also very important: every finite-dimensional representation decomposes as a direct sum of indecomposables, uniquely up to reordering. We can describe the indecomposable projectives fairly easily as well; the full list of indecomposables is more complicated, see [KJ16].

Definition 2.10 ($P(i)$): Define the representations $P(i)$ for $i \in I$ to be the $\overrightarrow{Q}$-representation associated to the left $\mathbb{C}\overrightarrow{Q}$-module $(\mathbb{C}\overrightarrow{Q})e_i$, spanned by all paths starting at i.

Note that the $P(i)$ are clearly projective, as $\mathbb{C}\overrightarrow{Q} = \bigoplus_{i \in I} P(i)$. They're characterized by the fact that for any $\overrightarrow{Q}$-representation V, we have $\operatorname{Hom}_{\overrightarrow{Q}}(P(i), V) = V_i$.

Theorem 2.11: Assume $\overrightarrow{Q}$ is acyclic. Then $\{P(i) \mid i \in I\}$ are the full list of nonzero projective indecomposables in $\operatorname{Rep} \overrightarrow{Q}$.

2.4 Grothendieck group

Definition 2.12 ($K(\overrightarrow{Q})$): Let $K(\overrightarrow{Q}) \coloneqq K(\operatorname{Rep} \overrightarrow{Q})$, the Grothendieck group of the abelian category $\operatorname{Rep} \overrightarrow{Q}$.

Definition 2.13 (graded dimension): Define the **graded dimension** $\mathbf{dim}\, V \in \mathbb{Z}^I$ to be the $|I|$-tuple given by $(\mathbf{dim}\, V)_i = \dim V_i$.

Theorem 2.14: Let $\overrightarrow{Q}$ be acyclic. Then the graded dimension map $\mathbf{dim}$ induces an isomorphism $K(\overrightarrow{Q}) \xrightarrow{\sim} \mathbb{Z}^I$.

Definition 2.15: Define the number $\langle V, W \rangle$ of two $\overrightarrow{Q}$-representations V, W to be

$$\langle V, W \rangle \coloneqq \sum_i (-1)^i \dim \operatorname{Ext}^i(V, W) = \chi(\mathbf{R}\operatorname{Hom}(V, W)).$$

Remark 2.16: It's known that we can always take a two-step projective resolution of any $\overrightarrow{Q}$-representation, hence the category $\operatorname{Rep} \overrightarrow{Q}$ is **hereditary**, i.e. all $\operatorname{Ext}^{>1}$ vanish.

It turns out that the Euler form is *very* insensitive to the representation itself.

Theorem 2.17: The number $\langle V, W \rangle$ depends only on the graded dimensions of the representations, and hence descends to a bilinear form on $\mathbb{Z}^I$, called the **Euler form**. In fact, for $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^I$,

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i \in I} \mathbf{v}_i \cdot \mathbf{w}_i - \sum_{e: i \to j \in \Omega} \mathbf{v}_i \mathbf{w}_j.$$

Note that the Euler form is not symmetric, so we'll frequently use the **symmetrized Euler form**

$$(\mathbf{v}, \mathbf{w}) \coloneqq \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle.$$

Remark 2.18: Note that the symmetrized Euler form is independent of the orientation of $\overrightarrow{Q}$.

3 Stability conditions

Fix a finite acyclic quiver $\overrightarrow{Q}$. We want to study stability conditions on $\operatorname{Rep}\overrightarrow{Q}$.

3.1 Moduli space of $\overrightarrow{Q}$-representations

In order to discuss stability conditions on $\overrightarrow{Q}$-representations, we need to enumerate all isomorphism classes of them. We know that the class of a $\overrightarrow{Q}$-representation V depends only on its graded dimension $\mathbf{dim}\, V \in \mathbb{Z}^I$; however, there may be many isomorphism classes of such representations. So let's fix some $\mathbf{v} \in \mathbb{Z}^I$ and study all $\overrightarrow{Q}$-representations with graded dimension $\mathbf{v}$.

Let's consider what such a V would look like. We know that $\dim V_i = \mathbf{v}_i$, so $V_i = \mathbb{C}^{\mathbf{v}_i}$. It only remains to parametrize the morphisms between the V_i. So define

$$\mathcal{R}_{\mathbf{v}} \coloneqq \bigoplus_{e:i\to j\in\Omega} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{\mathbf{v}_i}, \mathbb{C}^{\mathbf{v}_j}).$$

However, each isomorphism class of representation appears many times; isomorphisms are given by invertible maps $V_i \xrightarrow{\sim} V_i$ for all $i \in I$, so we need to quotient by this. Define

$$\mathrm{GL}_{\mathbf{v}} \coloneqq \prod_{i\in I} \mathrm{GL}_{\mathbf{v}_i}.$$

Then $\mathrm{GL}_{\mathbf{v}}$ naturally acts on $\mathcal{R}_{\mathbf{v}}$ by conjugation:

$$(g_i)_{i\in I} \cdot (\varphi_e)_{e:i\to j\in\Omega} = (g_j \varphi_e g_i^{-1})_{e:i\to j}.$$

It's clear that

$$\{\mathrm{GL}_{\mathbf{v}}\text{-orbits in } \mathcal{R}_{\mathbf{v}}\} \longleftrightarrow \{\text{isomorphism classes of } \overrightarrow{Q}\text{-representations with graded dimension } \mathbf{v}\}.$$

So we need to make sense $\mathcal{R}_{\mathbf{v}}/\mathrm{GL}_{\mathbf{v}}$, or whatever is the appropriate analogue of that quotient in the world of varieties.

Remark 3.1: Since $\mathrm{GL}_{\mathbf{v}}$ acts by conjugation, its action factors through $\mathrm{PGL}_{\mathbf{v}} \coloneqq \mathrm{GL}_{\mathbf{v}}/\mathbb{C}^\times$ (as scalars act trivially), so we may freely replace $\mathrm{GL}_{\mathbf{v}}$ with $\mathrm{PGL}_{\mathbf{v}}$ throughout.

3.2 GIT quotients

Let G be a reductive algebraic group acting algebraically on an affine algebraic variety M. Of course, in our setup, we take $\mathrm{G} = \mathrm{GL}_{\mathbf{v}}$ and $M = \mathcal{R}_{\mathbf{v}}$. This subsection explains GIT quotients; we won't *really* need it for studying stability conditions on quiver representations, since the main focus is actually on *twisted* GIT quotients (see §3.3), but twisted GIT quotients are in some sense a generalization of GIT quotients, so this subsection may be helpful to the reader.

The idea is that we want to build a moduli space for the G-orbits in M, i.e., a scheme version of M/G (which is a perfectly reasonable topological space, but rarely has many useful properties beyond that). This is actually rather hard, because the orbits come in many varying sizes and shapes. If we want the moduli space to actually be a scheme (even an affine variety) so that we can do our best with constructing quotients in the category of schemes (or affine varieties), we'll need to compromise and give up a lot.

Definition 3.2 (GIT quotient): We define the **GIT quotient** $M /\!\!/ \mathrm{G} \coloneqq \operatorname{Spec} \mathbb{C}[M]^{\mathrm{G}}$.

This is supposed to be our scheme version of the topological quotient M/G. It is indeed a scheme, and even an affine variety (the algebra $\mathbb{C}[M]^{\mathrm{G}}$ is finitely-generated due to HIlbert). However, topologically, the points of $M /\!\!/ \mathrm{G}$ are only the **closed orbits** in M, not all orbits. There's a natural topological map $M/\mathrm{G} \longrightarrow M /\!\!/ \mathrm{G}$ (sending a G-orbit $\mathbb{O}$ to the maximal ideal in $\mathbb{C}[M]^{\mathrm{G}}$ of functions vanishing on $\mathbb{O}$). However, whenever two orbits $\mathbb{O}, \mathbb{O}' \in M/\mathrm{G}$ "intersect," i.e., $\overline{\mathbb{O}} \cap \overline{\mathbb{O}'} \neq \emptyset$, then they get identified in $M /\!\!/ \mathrm{G}$. One way to understand this is that the G-orbits form a stratification of M, hence there's a partial order on the orbits where $\mathbb{O} \leq \mathbb{O}' \longleftrightarrow \overline{\mathbb{O}} \subset \overline{\mathbb{O}'}$, and the closed orbits are exactly the minimal elements of this partial ordering. The GIT quotient thus only remembers the minimal elements, i.e., the closed orbits.

Example 3.3: Let $\mathbb{G}_m \curvearrowright \mathbb{A}^2$ by the standard scaling action on both coordinates. We have many orbits: namely, we have the unique closed orbit $\{(0,0)\}$, and then we have a ton of dimension one orbits indexed by the ratio $(a, b) \mapsto b/a$. However, the GIT quotient only cares about the closed orbits; here, there's only one, so the GIT quotient is $\operatorname{Spec} \mathbb{C}$, which is just a point. This can also be computed by checking the $\mathbb{G}_m$-invariants in $\mathbb{C}[x, y]$, for which we quickly find that there are none except the constants.

So the GIT quotient can lose quite a lot of information.

Example 3.4: In the case of $M = \mathcal{R}_{\mathbf{v}}$ and $\mathrm{G} = \mathrm{GL}_{\mathbf{v}}$ (or $\mathrm{PGL}_{\mathbf{v}}$), the closed points are just the semisimple representations, so the GIT quotient $\mathcal{R}_{\mathbf{v}} /\!\!/ \mathrm{PGL}_{\mathbf{v}}$ is the moduli space of semisimple representations.

Now if $\overrightarrow{Q}$ has no oriented cycles, for example if it's a Dynkin diagram, then $\mathcal{R}_{\mathbf{v}} /\!\!/ \mathrm{PGL}_{\mathbf{v}}$ is actually just a single point, as there's a unique semisimple representation for each dimension $\mathbf{v}$. So this doesn't always give us much information.

3.3 Twisted GIT quotient

Once again, let G be a reductive algebraic group acting algebraically on an affine algebraic variety M; in our setup, we take $\mathrm{G} = \mathrm{GL}_{\mathbf{v}}$ and $M = \mathcal{R}_{\mathbf{v}}$.

We will review the theory of twisted GIT quotients, which will actually be the relevant theory in our case. Let $\chi : \mathrm{G} \longrightarrow \mathbb{G}_m$ be a character. Define

$$\mathbb{C}[M]^{\mathrm{G},\chi} \coloneqq \{f \in \mathbb{C}[M] \mid f(g \cdot m) = \chi(g) \cdot f(m)\},$$

the **relative invariants**. We get a graded algebra

$$\bigoplus_{n \geq 0} \mathbb{C}[M]^{\mathrm{G},\chi^n},$$

and Hilbert's theorem implies that it is finitely generated.

Definition 3.5 (twisted GIT quotient): The **twisted GIT quotient** is defined to be

$$M /\!\!/_{\chi} \mathrm{G} \coloneqq \operatorname{Proj}\left(\bigoplus_{n \geq 0} \mathbb{C}[M]^{\mathrm{G},\chi^n}\right).$$

The 0th graded component, $\mathbb{C}[M]^{\mathrm{G},\chi^0} = \mathbb{C}[M]^{\mathrm{G}}$ recovers the standard GIT quotient $M /\!\!/ \mathrm{G}$. Thus we get a projective morphism

$$\pi : M /\!\!/_{\chi} \mathrm{G} \to M /\!\!/ \mathrm{G}. \tag{1}$$

3.4 GIT stability

We continue the setup as in §3.3.

Definition 3.6 (GIT (semi)stability): Extend the action of G on M to an action on $M \times \mathbb{A}^1$ by $g(m, z) \coloneqq (g(m), \chi^{-1}(g)z)$.
A point $x \in M$ is χ-**semistable** if for any nonzero $z \in \mathbb{C} - \{0\}$, the closure of the G-orbit of (x, z) is disjoint from the zero section $M \times \{0\}$. We denote the set of χ-semistable points of M by M^{ss}_χ.
A point $x \in M$ is χ-**stable** if it is χ-semistable, has finite stabilizer $\mathrm{G}_x \subset \mathrm{G}$, and for any nonzero z the G-orbit of (x, z) is closed in $M \times \mathbb{A}^1$. In fact, this is equivalent to the G-orbit of x being closed in M^{ss}_χ. We denote the set of χ-stable points of M by M^s_χ.

To describe these conditions more explicitly, we'll make frequent use of:

Theorem 3.7 (Geometric reductivity principle): If $X, Y \subset M$ are closed G-invariant subvarieties and $X \cap Y = \emptyset$, then there exists a G-invariant polynomial f such that $f|_X = 0$ and $f|_Y = 1$.

We immediately deduce a technical condition about χ-semistability.

Corollary 3.8: A point $x \in M$ is χ-semistable iff there exists $f \in \mathbb{C}[M]^{\mathrm{G},\chi^n}$ for some $n \geq 1$, for which $f(x) \neq 0$.

Proof. Suppose x is χ-semistable. We apply the geometric reductivity principle (3.7) to the G-action on $M \times \mathbb{A}^1$, which tells us there's a function $\widehat{f} \in \mathbb{C}[M \times \mathbb{A}^1]^{\mathrm{G}}$ such that $\widehat{f}|_{M \times \{0\}} = 0$ and $\widehat{f}|_{\overline{\mathrm{G} \cdot (x,1)}} \neq 0$. Since $\widehat{f} \in \mathbb{C}[M \times \mathbb{A}^1]^{\mathrm{G}}$, and G acts on the $\mathbb{A}^1$-component by χ^{-1}, we know that G must act correspondingly by χ on the M-coordinate; thus we can write

$$\widetilde{f}(x, z) = \sum_{n \geq 0} f_n(x) z^n, \quad f_n \in \mathbb{C}[M]^{\mathrm{G},\chi^n}.$$

Now by hypothesis $\widehat{f}|_{M \times \{0\}} = 0$, so $\widehat{f}(m, 0) = f_0(m) = 0 \implies f_0 = 0$. But since $\widehat{f}$ is not identically zero (it is nonzero on the closure of the G-orbit of $(x, 1)$), then there must be some $0 \neq f_n \in \mathbb{C}[M]^{\mathrm{G},\chi^n}$ with $f_n(x) \neq 0$.
In the other direction, if $f \in \mathbb{C}[M]^{\mathrm{G},\chi^n}$ is such that $f(x) \neq 0$, then the function $\widetilde{f} \coloneqq (x, z) \mapsto f(x) \cdot z^n$ is G-invariant on $M \times \mathbb{A}^1$. Since $f(x) \neq 0$, it's clear that for any $z \neq 0$, then $\widetilde{f}(x, z) \neq 0$, hence is a nonzero constant on the entire G-orbit $\mathrm{G} \cdot (x, z)$, hence is a nonzero constant on the closure $\overline{\mathrm{G} \cdot (x, z)}$ as well. But $\widetilde{f}(m, 0) = f(m) \cdot 0^n = 0$, so $\widetilde{f}|_{M \times \{0\}} = 0$. It follows that $\overline{\mathrm{G} \cdot (x, z)} \cap M \times \{0\} = \emptyset$, so $x \in M^{ss}_\chi$. □

Corollary 3.9:

a) $M^{ss}_\chi \subset M$ is open and G-invariant (but possibly empty).

b) For $N \in \mathbb{Z}_{>0}$, $x \in M$ is χ-semistable iff it is χ^N-semistable. Thus, the notion of χ-semistable can be defined for any rational character $\chi \in X(\mathrm{G}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

c) Every $x \in M^{ss}_\chi$ defines a maximal ideal $J_x \coloneqq \{f \mid f(x) = 0\} \subset \bigoplus_{n \geq 0} \mathbb{C}[M]^{\mathrm{G},\chi^n}$, and is not the irrelevant ideal. Thus we have a natural map

$$M^{ss}_\chi / \mathrm{G} \longrightarrow M /\!\!/_\chi \mathrm{G}, \quad x \mapsto J_x.$$

In lieu of this, it would be nice to understand the map $M^{ss}_\chi / \mathrm{G} \longrightarrow M /\!\!/_\chi \mathrm{G}$.

Theorem 3.10:

a) The map $M^{ss}_\chi / \mathrm{G} \longrightarrow M /\!\!/_\chi \mathrm{G}$ is surjective.

b) Two points $x, y \in M^{ss}_\chi / \mathrm{G}$ (corresponding to semistable G-orbits $\mathbb{O}_x, \mathbb{O}_y \subset M^{ss}_\chi$) are mapped to the same point in $M /\!\!/_\chi \mathrm{G}$ iff the closures of their orbits (taken in M^{ss}_χ) intersect, i.e., $\overline{\mathbb{O}_x} \cap \overline{\mathbb{O}_y} \cap M^{ss}_\chi \neq \emptyset$.

c) As a topological space, $M /\!\!/_\chi \mathrm{G} = M^{ss}_\chi / \sim$, where $x \sim y$ iff the closures of their orbits in M^{ss}_χ intersect.

d) In fact, $M /\!\!/_\chi \mathrm{G} = \{\text{closed orbits in } M^{ss}_\chi\}$. (Note that this is weaker than being closed in M.)

Using this explicit description of $M /\!\!/_\chi \mathrm{G}$, we can explicitly describe the map $\pi : M /\!\!/_\chi \mathrm{G} \longrightarrow M /\!\!/ \mathrm{G}$ from (1).

Theorem 3.11: Let $x \in M_\chi^{ss}$ and denote $[x]$ its image in $M /\!\!/_\chi \mathrm{G}$; recall that every point in $M /\!\!/_\chi \mathrm{G}$ is of the form $[x]$ for $x \in M_\chi^{ss}$, and also that we have a map $\pi : M /\!\!/_\chi \mathrm{G} \to M /\!\!/ \mathrm{G}$. Then

$$\pi([x]) = \text{the unique closed orbit in } M \text{ contained in } \overline{\mathbb{O}_x}.$$

Proof. Let $\mathbb{O}_1$ be the unique closed orbit in $\overline{\mathbb{O}_x}$. For $f \in \mathbb{C}[M]^{\mathrm{G}}$, we can verify that $f(\mathbb{O}_1) = \pi^* f(\mathbb{O}_x)$. □

So more or less, what's happening is that when you take a GIT quotient, you form a partial ordering on the orbits (by containment of the closure of the orbits); the closed orbits are the minimal ones, and the GIT quotient only remembers the minimal ones. So the GIT quotient $M /\!\!/ \mathrm{G}$ remembers only the smallest G-orbits in M. But the twisted GIT quotient $M /\!\!/_\chi \mathrm{G}$ only requires that the orbits are closed in $M_\chi^{ss} \subset M$; this is weaker than being closed in M, and the map $\pi : M /\!\!/_\chi \mathrm{G} \to M /\!\!/ \mathrm{G}$ "remembers" the rest of the orbit as we add back the complement $M \setminus M_\chi^{ss}$, and then sends the closed-in-M_χ^{ss}-but-not-in-M orbits to the true minimal closed orbit contained in its closure.

Remark 3.12: In our specific case, letting $M = \mathcal{R}_{\mathbf{v}}$, then the map π send a representation V to its *semisimplification* V^{ss}. The semisimplification of V is defined basically in the only way possible: take a Jordan-Holder filtration of V, and define V^{ss} to be the direct sum of the (simple) subquotients, thus turning it into a semisimple representation of the same graded dimension. (Yes, this is bad notation; V^{ss} means semisimplification, while M^{ss} means semistable, but it's not my fault both words are just two s-words put together.)

We'd like to say things about stable points as well. Recall that the property of being stable implies that their G-orbits intersect iff their closures intersect in M_χ^{ss}, hence by Theorem 3.10c, distinct stable orbits define distinct points in $M /\!\!/_\chi \mathrm{G}$, so

$$M_\chi^s \subset M /\!\!/_\chi \mathrm{G}.$$

Theorem 3.13: Assume $M_\chi^s \neq \emptyset$.

a) M_χ^s is open in M_χ^{ss}, and thus in M.

b) If M is irreducible (which it will be for us - it'll be $\mathcal{R}_{\mathbf{v}}$), then M_χ^s is dense in M_χ^{ss}, and M_χ^s/G is dense in $M /\!\!/_\chi \mathrm{G}$.

c) If M is nonsingular (again, it will be for us) and for every $x \in M_\chi^s$, the stabilizer G_x is trivial, then M_χ^s/G is a nonsingular variety of dimension $\dim M - \dim \mathrm{G}$.

Finally, we'll also make note of a numerical criterion which detects (semi)stability.

Theorem 3.14 (Mumford): A point $x \in M$ is semistable (respectively stable), iff for any one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{G}$ such that $\lim_{t\to 0} \lambda(t) \cdot x$ exists, then $\langle \chi, \lambda \rangle \geq 0$ (respectively, $\langle \chi, \lambda \rangle > 0$, for nontrivial λ).

3.5 Classical stability

Our lattice will be $K(\overrightarrow{Q}) \simeq \mathbb{Z}^I$. We first need to fix a linear functional $\theta : \mathbb{R}^I \to \mathbb{R}$.

Definition 3.15 (slope): Define the θ-**slope** of a $\overrightarrow{Q}$-representation V to be

$$\mu_\theta(V) = \frac{\theta(\mathbf{dim}\, V)}{\dim V},$$

where $\dim V \coloneqq \sum_i (\mathbf{dim}\, V)_i$ is the total dimension of the vector spaces.

Definition 3.16 (classical (semi)stability): A representation V is **(classically)** μ**-semistable** if for every proper nonzero submodule $M \subset V$, then $\mu(M) \leq \mu(V)$. It is **stable** if additionally $\mu(M) < \mu(V)$.

Remark 3.17: This classical notion of (semi)stability is analogous to the classical notion of (semi)stable sheaves on smooth projective varieties.

3.6 The stability conditions agree

Fix $\theta = (\theta_i)_{i \in I}$ a linear functional on $\mathbb{Z}^I$, and define μ_θ as above. Define the character of $\mathrm{GL}_\mathbf{v}$ by

$$\chi_\theta : \quad \mathrm{GL}_\mathbf{v} \ni (g_i)_{i \in I} \mapsto \prod_{i \in I} \mapsto \det(g_i)^{\mu_\theta(\mathbf{v}) - \theta_i} \in \mathbb{C}^\times.$$

We have two notions of a representation V of graded dimension $\mathbf{v}$ being (semi)stable: one from the GIT sense, and one from the classical sense.

Theorem 3.18: A $\overrightarrow{Q}$-representation V of graded dimension $\mathbf{v}$ is χ_θ-(semi)stable in the GIT sense (as a point in $\mathcal{R}_\mathbf{v}$) iff it is μ_θ-(semi)stable in the classical sense.

Proof. We'll just prove it for semistable; the proof for stable is exactly the same, but replacing all of the $\leq$ with $<$. The key is to leverage Mumford's criterion (3.14) on the GIT side with filtrations on the classical side, so we need to understand how one-parameter subgroups interact with filtrations.

Lemma 3.19: Fix $V \in \mathcal{R}_\mathbf{v}$, a $\overrightarrow{Q}$-representation such that $\mathbf{dim}\, V = \mathbf{v}$. Let $V = \left(\{V_i\}_{i \in I}, \{\varphi_e\}_{e \in \Omega}\right)$.
To a one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{GL}_\mathbf{v}$ such that $\lim_{t \to 0} \lambda(t)$ exists, we obtain a finite filtration of V by subrepresentations. Conversely, to each (necessarily finite) filtration of V by subrepresentations, we obtain (non-uniquely) a one-parameter subgroup λ such that $\lim_{t \to 0} \lambda(t)$ exists.

Remark 3.20: We are not claiming that these are inverse operations; however, they are inverses in one direction: to a filtration of V, we produce a one-parameter subgroup λ whose limit exists, and the filtration we obtain from λ recovers our original filtration. The failure of the reverse composition is due to the choice of direct summand complement, so there are many one-parameter subgroups we could choose inducing the same filtration.

Proof. First suppose we have a one-parameter subgroup λ. We already have an action $\mathrm{GL}_\mathbf{v} \curvearrowright \mathcal{R}_\mathbf{v}$, hence λ induces an action of $\mathbb{G}_m$ on each V_i, $i \in I$. But a $\mathbb{G}_m$-action is the same as a $\mathbb{Z}$-grading, hence each V_i decomposes as $\bigoplus_{n \in \mathbb{Z}} V_i^{(n)}$, where $\lambda(t)|_{V_i^{(n)}} = t^n$. Write $V_i^{\geq n} \coloneqq \bigoplus_{m \geq n} V_i^{(m)}$.
Now for each edge $e : i \to j \in \Omega$, the linear map $\varphi_e : V_i \to V_j$ decomposes into a direct sum $\varphi_e^{m,n} : V_i^{(n)} \to V_j^{(m)}$, with action of $\lambda(t)$ by

$$\lambda(t) \cdot \varphi_e^{m,n} = \lambda(t)|_{V_j} \cdot \varphi_e^{m,n} \cdot \lambda(t)|_{V_i}^{-1} = t^m \cdot \varphi_e^{m,n} \cdot t^{-n} = t^{m-n} \varphi_e^{m,n}.$$

So the limit $\lim_{t \to 0} \lambda(t)$ existing implies that for $m < n$, we have $\varphi_e^{m,n} = 0$, otherwise the λ-action blows $\varphi_e^{m,n}$ up to infinity. Thus φ_e always **increases** the weights (of the λ-action), hence we have well-defined maps $\varphi_e : V_i^{\geq n} \to V_j^{\geq n}$ for all $e \in \Omega$, and thus $V^{\geq n} \coloneqq \left(\left\{V_i^{\geq n}\right\}_{i \in I}, \{\varphi_e\}_{e \in \Omega}\right)$ defines a subrepresentation. Thus from λ we obtain a filtration $\cdots \subseteq V^{\geq n+1} \subseteq V^{\geq n} \subseteq V^{\geq n-1} \subseteq \cdots$ of V by subrepresentations, and it must be finite because V is finite-dimensional.
On the other hand, let's suppose we have some finite filtration $V = V^k \supseteq V^{k+1} \supseteq \cdots \supseteq V^{k+n} = 0$ of V by subrepresentations. Then we can artificially construct a one-parameter subgroup (whose limit exists) by choosing some direct summand complement to each V^{i+1} in V^i, and declaring that $\lambda(t)$ acts on this direct summand complement by t^i. □

We also need to know one more thing: what $\langle \chi_\theta, \lambda \rangle$ is.

Lemma 3.21: Fix some V as before. Let λ be a one-parameter subgroup whose limit exists; by Lemma 3.19, we get an induced filtration by $V^{\geq n}$. Then $\langle \chi_\theta, \lambda \rangle = \sum_{n \in \mathbb{Z}} \left(\dim(V^{\geq n}) \mu_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, V^{\geq n})\right)$.

Proof. We can compute the composition $\chi_\theta \circ \lambda$ directly:

$$\chi_\theta(\lambda(t)) = \prod_{i\in I} \det(\lambda(t)_i)^{\mu_\theta(\mathbf{v})-\theta_i} = \prod_{i\in I}\prod_{n\in\mathbb{Z}} \det\left(\lambda(t)|_{V_i^{(n)}}\right)^{\mu_\theta(\mathbf{v})-\theta_i} = \prod_{i\in I}\prod_{n\in\mathbb{Z}} t^{n\cdot(\dim V_i^{(n)})\cdot(\mu_\theta(\mathbf{v})-\theta_i)}.$$

This computation tells us $\langle \chi_\theta, \lambda\rangle$:

$$\begin{aligned}
\langle \chi_\theta, \lambda\rangle &= \sum_{i\in I}\sum_{n\in\mathbb{Z}} n\cdot(\dim V_i^{(n)})\cdot(\mu_\theta(\mathbf{v})-\theta_i),\\
&= \sum_{n\in\mathbb{Z}} n\cdot\left(\dim(V^{\geq n}/V^{\geq n+1})\chi_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, V^{\geq n}/V^{\geq n+1})\right),\\
&= \sum_{n\in\mathbb{Z}} n\left(\dim(V^{\geq n})\mu_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, V^{\geq n})\right) - n\left(\dim(V^{\geq n+1})\mu_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, V^{\geq n+1})\right),\\
&= \sum_{n\in\mathbb{Z}} \left(\dim(V^{\geq n})\mu_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, V^{\geq n})\right).
\end{aligned}$$

□

Now let's return to the proof. Suppose V is χ_θ-semistable (in the GIT sense). We want to show that it is μ_θ-semistable (in the classical sense). So let $M \subset V$ be any proper nonzero subrepresentation, and treat this as the (very short) filtration $0 \subset M \subset V$. Then using Lemma 3.19, we can construct some one-parameter subgroup λ. Since V is χ_θ-semistable, Mumford's criterion (3.14) implies that $0 \leq \langle\chi_\theta,\lambda\rangle$. But in Lemma 3.21, we compute that

$$\begin{aligned}
&0 \leq \langle\chi_\theta,\lambda\rangle,\\
&\quad = \dim(0)\mu_\theta(\mathbf{v}) - \theta(0) + \dim(M)\mu_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, M) + \dim(V)\mu_\theta(\mathbf{v}) - \theta(\mathbf{v}),\\
&\quad = \dim(M)\mu_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, M),\\
\implies\ &\mu_\theta(M) \leq \mu_\theta(\mathbf{v}) = \mu_\theta(V),
\end{aligned}$$

so we conclude that V being χ_θ-semistable implies V is μ_θ-semistable.
Conversely, suppose V is μ_θ-semistable. To show that V is χ_θ-semistable, we just need to show that $\langle\chi_\theta,\lambda\rangle \geq 0$ for every λ whose limit exists. For any such λ, Lemma 3.19 gives us a filtration of V by subrepresentations $V^{\geq n}$. Then since V is μ_θ-semistable, we must have

$$\mu_\theta(V^{\geq n}) \leq \mu_\theta(V) = \mu_\theta(\mathbf{v})$$

for all n; this implies that

$$\dim(V^{\geq n})\mu_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, V^{\geq n}) \geq 0.$$

Then Lemma 3.21 computes that

$$\langle\chi_\theta,\lambda\rangle = \sum_{n\in\mathbb{Z}} \left(\dim(V^{\geq n})\mu_\theta(\mathbf{v}) - \theta(\mathbf{dim}\, V^{\geq n})\right) \geq 0,$$

hence Mumford's criterion (3.14) implies that V is χ_θ-semistable. □

3.7 A worked-out example

Let's consider the quiver A_2:

$$A_2 \coloneqq \bullet \longrightarrow \bullet.$$

There are two vertices, hence two simple representations, and so

$$K(A_2) \simeq \mathbb{Z}^2.$$

There are exactly **three** indecomposable representations, up to isomorphism:

- V_1: $\mathbb{C} \to 0$.

- V_2: $0 \longrightarrow \mathbb{C}$.
- V_3: $\mathbb{C} \xrightarrow{\text{id}} \mathbb{C}$.

Note that V_1 and V_2 are the simple representations associated to the vertices, see (2.9).

Remark 3.22: In this case, A_2 is what's known as a *Dynkin quiver*, in that its underlying (undirected) graph is a Dynkin diagram. It corresponds to the simple Lie algebra $\mathfrak{sl}_3$, and it's known that the indecomposable representations are in bijection with the positive roots of $\mathfrak{sl}_3$, of which it has three. Furthermore, to a positive root $\alpha = \sum_{i \in I} n_i \alpha_i$, where α_i are the simple roots, the associated indecomposable representation has graded dimension $(n_i)_{i \in I}$. In our case, there are three positive roots: the simple roots α_1 and α_2 (which must have graded dimension $(1,0)$ and $(0,1)$, respectively), and the positive root $\alpha_1 + \alpha_2$ which has graded dimension $(1,1)$.

Let us fix our graded dimension to be $\mathbf{v} = (1,1)$, so that $\mathrm{GL}_{\mathbf{v}} = \mathrm{GL}_1 \times \mathrm{GL}_1 = \mathbb{C}^\times \times \mathbb{C}^\times$. We can easily see that there are exactly two representations of graded dimension $\mathbf{v}$: these are $V_1 \oplus V_2$ and V_3. Let us take $\theta = (a, b) \in \mathbb{Z}^2$ some arbitrary linear functional, and let's study when these representations are (semi)stable.

Example 3.23 (semistability for V_3): First, let's examine the **classical** case. First, we compute:

$$\mu_\theta(V_3) = \frac{\theta((1,1))}{1+1} = \frac{a+b}{2}.$$

Now the only subrepresentation of V_3 is V_2, so we need to check that $\mu_\theta(V_2) \leq \mu_\theta(V_3)$. We have

$$\mu_\theta(V_2) = \frac{\theta((0,1))}{0+1} = b,$$

so

$$V_3 \text{ is } \mu_\theta\text{-semistable} \iff \mu_\theta(V_2) \leq \mu_\theta(V_3) \iff b \leq a.$$

Now let's look at the **GIT** side. Our character is

$$\chi_\theta : \quad \mathbb{C}^\times \times \mathbb{C}^\times \ni (s,t) \mapsto g^{\frac{a+b}{2} - a} \cdot s^{\frac{a+b}{2} - b} = \left(\frac{s}{g}\right)^{\frac{a-b}{2}}.$$

Now a one-parameter subgroup $\lambda : \mathbb{C}^\times \to \mathbb{C}^\times \times \mathbb{C}^\times$ is just a product of two characters, $t \mapsto (t^m, t^n)$; so take $\lambda = (m, n)$. So we started with $V_3 = \left(\mathbb{C} \xrightarrow{\cdot 1} \mathbb{C}\right)$; we compute that

$$\lambda(t) \cdot V_3 = \left(\mathbb{C} \xrightarrow{\cdot t^{n-m}} \mathbb{C}\right).$$

It follows that the limit $\lim_{t \to 0} \lambda(t) \cdot V_3$ exists iff $n \geq m$; so we need only consider one-parameter subgroups λ corresponding to (n, m) with $n \geq m$. Now we just compute that

$$\chi_\theta \circ \lambda : \quad t \mapsto \left(\frac{t^n}{t^m}\right)^{\frac{a-b}{2}} = t^{(n-m)(a-b)/2}.$$

Then

$$V_3 \text{ is } \chi_\theta\text{-semistable} \iff \langle \lambda_{m,n}, \chi_\theta \rangle \geq 0 \text{ for all } n \geq m \iff \frac{(n-m)(a-b)}{2} \geq 0 \text{ for all } n \geq m \iff a \geq b.$$

So we conclude that the two notions of stability are indeed exactly the same here.

Example 3.24 (stability for V_3): Running through the previous argument, we have

$$V_3 \text{ is } \mu_\theta\text{-stable} \iff a > b,$$

and

$$V_3 \text{ is } \chi_\theta\text{-stable} \iff \frac{(n-m)(a-b)}{2} > 0 \text{ for all } n > m \iff a > b.$$

So once again, they agree. (Note that this time, we need to use the modified version of Mumford's criterion (3.14), which requires λ to be *nontrivial*, which is equivalent to $n > m$.)

Example 3.25 (semistability for $V_1 \oplus V_2$): Let's again start with the classical case. We have two subrepresentations of $V_1 \oplus V_2$, namely V_1 and V_2. Then $\mu_\theta(V_1) = a$, $\mu_\theta(V_2) = b$, and $\mu_\theta(V_1 \oplus V_2) = \frac{a+b}{2}$. Therefore

$$V_1 \oplus V_2 \text{ is } \mu_\theta\text{-semistable} \iff \mu_\theta(V_1), \mu_\theta(V_2) \le \mu_\theta(V_1 \oplus V_2) \iff a = b.$$

Now let's look at the GIT side. Note that we started with

$$V_1 \oplus V_2 = \left(\mathbb{C} \xrightarrow{0} \mathbb{C}\right),$$

so for *any* one-parameter subgroup λ, then $\lambda(t)$ does nothing to $V_1 \oplus V_2$. Therefore the limit always exists, and so

$$V_1 \oplus V_2 \text{ is } \chi_\theta\text{-semistable} \iff 0 \le \langle \lambda, \chi_\theta \rangle \text{ for all } \lambda \iff \frac{(n-m)(a-b)}{2} \ge 0 \text{ for all } n, m \iff a = b.$$

So once again, the two notions agree.

Example 3.26 (stability for $V_1 \oplus V_2$): To be μ_θ semistable, we'd need both $a > b$ *and* $b > a$, which is impossible, so actually $V_1 \oplus V_2$ is **never** μ_θ-semistable (for any θ).
On the other hand, since every one-parameter subgroup acts trivially, $V_1 \oplus V_2$ cannot be χ_θ-semistable.

3.8 Moduli space of (semi)stable representations

Recall that $\mathcal{R}_\mathbf{v}$ is the moduli space of $\overrightarrow{Q}$-representations of dimension $\mathbf{v}$, albeit not up to isomorphism. We'll now study moduli spaces of θ-(semi)stable representations (of dimension $\mathbf{v}$).

Definition 3.27: For $\theta \in \mathbb{R}^I$, define

$$\mathcal{R}_\theta(\mathbf{v}) := \mathcal{R}_\mathbf{v} /\!\!/_{\chi_\theta} \mathrm{PGL}_\mathbf{v}.$$

Note that this is the moduli space of θ-semistable representations of dimension $\mathbf{v}$, semisimple with respect to χ_θ.

In the special case of $\theta = 0$, then $\mathcal{R}_0 = \mathcal{R}_v /\!\!/ \mathrm{PGL}_\mathbf{v}$ is legitimately just the moduli space of semisimple representations of dimension $\mathbf{v}$. Now recall that we have a map (see Theorem 3.11)

$$\pi : \mathcal{R}_\theta(\mathbf{v}) \to \mathcal{R}_0(\mathbf{v}),$$

sending V to its semisimplification V^{ss} (or more generally, sending a θ-semistable orbit $\mathbb{O}$ to the unique closed orbit contained in $\overline{\mathbb{O}}$).

Lemma 3.28: Let $\theta \in \mathbb{R}^I$. The category $\mathrm{Rep}_\theta^{ss}(\overrightarrow{Q})$ of θ-semistable representations of $\overrightarrow{Q}$ is an abelian category, and the simple objects are exactly θ-stable representations.

Proof sketch. The proof is fairly straightforward, so we'll leave out most of the details. To show it's an abelian category, we just need to show that kernels and images of morphisms between θ-semistable representations are still θ-semistable, which is easy to check by the definition of θ-semistability. The simple objects are easily seen to be θ-stable as they cannot have subobjects which are also θ-semistable. □

In particular, for any θ-semistable representation, we can define its θ-**semisimplification**, sending it to the direct sum of its composition factors (i.e., θ-stable subquotients) in the category $\mathrm{Rep}_\theta^{ss}(\overrightarrow{Q})$. A similar description of the orbits in $(\mathcal{R}_\mathbf{v})^{ss}_{\chi\theta}$ exist; see [KJ16, Theorem 10.7]. Let us now see what happens to the θ-stable points.

Theorem 3.29:
(a) $\mathrm{PGL}_\mathbf{v}$ acts freely on the set of θ-stable points of $\mathcal{R}_\mathbf{v}$.
(b) The quotient space

$$\mathcal{R}^s_\theta(\mathbf{v}) \coloneqq (\mathcal{R}_\mathbf{v})^s_{\chi\theta}/\mathrm{PGL}_\mathbf{v} \subset \mathcal{R}_\theta(\mathbf{v})$$

is a smooth subset. If it is nonempty, then it is open and dense, with dimension $1 - \langle \mathbf{v}, \mathbf{v}\rangle$ (the Euler form of $\overrightarrow{Q}$, see Theorem 2.17).

Unfortunately, this does not always give us much useful information. We already saw that $\mathcal{R}_0$ is just a point. In fact generally speaking $\mathcal{R}^s_\theta(\mathbf{v})$ is "usually" empty.

Example 3.30: Let $\overrightarrow{Q}$ be an acyclic quiver. Then

$$\left\{\exists\,\theta \text{ s.t. } \mathcal{R}^s_\theta(\mathbf{v}) \neq \emptyset\right\} \iff \{\text{for generic } x \in \mathcal{R}_\mathbf{v}, \quad \mathrm{End}(V_x) = \mathbb{C}\}\,.$$

If $\overrightarrow{Q}$ is a Dynkin quiver, the right side condition holds iff $\mathbf{v}$ is a root; so if $\mathbf{v}$ is not a root, then $\mathcal{R}^s_\theta(\mathbf{v}) = \emptyset$ for all θ. (In fact even when $\mathbf{v}$ is a root, $\mathcal{R}^s_\theta(\mathbf{v})$ is still either empty or a point.)

This will be corrected by the notion of framings.

4 Quiver varieties

We'll need several notions before we end up at the definition of the quiver variety.

4.1 Double quiver

Definition 4.1 (double quiver): Let Q be an undirected graph (for example, the undirected graph of a quiver). We define the corresponding **double quiver** $Q^\#$ to be the quiver resulting from keeping the vertices of Q, and then each edge in Q gives rise to two, oppositely-oriented edges in $Q^\#$.

In other words, just double every edge and set them in the opposite directions.

The set of edges of $Q^\#$ has a natural involution $(h : i \to j) \mapsto (\overline{h} : j \to i)$ which just reverses every edge orientation. Let the edges of $Q^\sharp$ be denoted by H; if $Q^\#$ arose from a quiver $\overrightarrow{Q}$, then $H = \Omega \cup \overline{\Omega}$, where $\overline{\Omega}$ is the edges of $\overrightarrow{Q}$ but inverted in direction. It's easy to see we can also choose some splitting $H = \Omega \cup \overline{\Omega}$, so that $Q^\#$ is realized as the double quiver of an oriented quiver $\overrightarrow{Q}$ (with edges Ω).

Example 4.2: Here is an example of a double quiver.

$$\bullet \rightleftarrows \bullet \rightleftarrows \bullet \rightleftarrows \bullet$$

The category of representations of a double quiver is defined in the same way as for any quiver. From §3.1, the isomorphism classes of representations of graded dimension $\mathbf{v}$ are in bijection with $\mathrm{GL}_\mathbf{v}$-orbits in the representation space

$$\mathcal{R}(Q^\#, \mathbf{v}) \coloneqq \bigoplus_{h:i\to j\in H} \mathrm{Hom}_\mathbb{C}(\mathbb{C}^{\mathbf{v}_i}, \mathbb{C}^{\mathbf{v}_j}).$$

Notice that H can be split into $\Omega \cup \overline{\Omega}$, so that for each $h : i \to j \in \Omega$, we have $\overline{h} : j \to i \in \overline{\Omega}$; this identifies $Q^\#$ as coming from a quiver $\overline{Q}$ (with oriented edges Ω). In other words for each $\mathrm{Hom}_{\mathbb{C}}(V, W)$ we also have $\mathrm{Hom}_{\mathbb{C}}(W, V) \simeq \mathrm{Hom}_{\mathbb{C}}(V, W)^*$. Using this, we can write

$$\mathcal{R}(Q^\#, \mathbf{v}) \simeq \mathcal{R}(\overrightarrow{Q}, \mathbf{v}) \oplus \mathcal{R}(\overrightarrow{Q}, \mathbf{v})^* \simeq T^*(\mathcal{R}(\overrightarrow{Q}, \mathbf{v})),$$

using the fact that $\mathcal{R}(\overrightarrow{Q}, \mathbf{v})$ is an affine space and the cotangent bundle of a vector space V is just $T^*V \simeq V \oplus V^*$.

4.2 Preprojective algebra

In general, classifying representations of double quivers is very hard, and generally doesn't have a good answer. We will instead consider representations satisfying additional constraints, for which classification is a tractable problem.

Definition 4.3 (preprojective algebra): Let $Q^\#$ be the double quiver of a quiver $\overrightarrow{Q}$, so that $H = \Omega \cup \overline{\Omega}$. Define the **preprojective algebra**

$$\Pi = \Pi(\overrightarrow{Q}) \coloneqq \mathbb{C}Q^\# / J,$$

where J is the two-sided ideal generated by elements Θ_i for $i \in I$ (running over all vertices) defined by

$$\Theta_i \coloneqq \sum_{e \in \Omega,\, e_h = i} \overline{e}e - \sum_{e \in \Omega,\, e_t = i} e\overline{e}.$$

(In other words, we take the signed sum over all length-two loops coming out of vertex i.)

We point out that up to isomorphism, the preprojective algebra does not depend on the choice of a subquiver $\overrightarrow{Q}$ generating it; in fact, we can define it more generally depending on some parameter function $\varepsilon : H \to \mathbb{C}^\times$ (see [KJ16]) but we won't need this generality. Additionally, for any choice of orientation Ω of $Q^\#$ (i.e. choosing a quiver $\overrightarrow{Q}$ giving rise to the double quiver $Q^\#$), the path algebra $\mathbb{C}\overrightarrow{Q}$ is a subalgebra of Π.

It is also easy to see from the definition (and that J is a homogeneous ideal) that the **preprojective algebra is naturally graded**, with the degree ℓ part Π^ℓ (for $\ell \geq 0$) being the span of paths of length ℓ, so that

$$\Pi^\ell = \bigoplus_{i,j \in I} {}_j\Pi_i^\ell$$

where ${}_j\Pi_i^\ell = e_j \Pi^\ell e_i$ is the span of paths of length ℓ from i to j.

There is also a variant we will need:

Definition 4.4 (deformed preprojective algebra): Let $Q^\#$ be the double quiver of a quiver $\overrightarrow{Q}$, so that $H = \Omega \cup \overline{\Omega}$. Define the **deformed preprojective algebra with parameter**

$$\lambda \coloneqq (\lambda_i)_{i \in I},$$

to be the algebra

$$\Pi_\lambda = \Pi_\lambda(\overrightarrow{Q}) \coloneqq \mathbb{C}Q^\# / J_\lambda,$$

where J_λ is the two-sided ideal generated by elements $\Theta_{i,\lambda}$ for $i \in I$ (running over all vertices) defined by

$$\Theta_{i,\lambda} \coloneqq \sum_{e \in \Omega,\, e_h = i} \overline{e}e - \sum_{e \in \Omega,\, e_t = i} e\overline{e} - \sum_{i \in I} \lambda_i \cdot e_i.$$

4.3 Moment map

Recall that for a choice of orientation Ω of an undirected graph Q, we get a quiver $\overrightarrow{Q}$ and a double quiver $Q^\#$. The edges H of $Q^\#$ can then be partitioned into $\Omega \cup \overline{\Omega}$, giving us an isomorphism

$$\mathcal{R}(Q^\#, \mathbf{v}) \simeq \mathcal{R}(\overrightarrow{Q}, \mathbf{v}) \oplus \mathcal{R}(\overrightarrow{Q}, \mathbf{v})^* \simeq T^*(\mathcal{R}(\overrightarrow{Q}, \mathbf{v})).$$

Now any cotangent space has a canonical symplectic form ω. Explicitly, we can write it as

$$\omega((x_1, y_2), (x_2, y_2)) = \langle y_1, x_2 \rangle - \langle y_2, x_1 \rangle$$

where $x_i \in \mathcal{R}(\overrightarrow{Q}, \mathbf{v})$ and $y_i \in \mathcal{R}(\overrightarrow{Q}, \mathbf{v})^*$. (Once again, there's a more general notion depending on some function $\varepsilon : H \longrightarrow \mathbb{C}^\times$, but it doesn't really change anything, so we won't discuss it.)

Notice that the action of $\mathrm{GL}_\mathbf{v} \curvearrowright \mathcal{R}(Q^\#, \mathbf{v})$ preserves the form ω. This leads us to:

Theorem 4.5: The action of $\mathrm{GL}_\mathbf{v}$ on $\mathcal{R}(Q^\#, \mathbf{v})$ is Hamiltonian. The corresponding moment map

$$\mu_\mathbf{v} : \mathcal{R}(Q^\#, \mathbf{v}) \to \bigoplus_i \mathfrak{gl}_{\mathbf{v}_i}$$

is given by

$$z \mapsto \sum_{h \in \Omega} [z_h, z_{\overline{h}}].$$

Remark 4.6: Note that $\mathfrak{gl}_{\mathbf{v}_i} \simeq \mathrm{End}(\mathbb{C}^{\mathbf{v}_i})$. Also, here we identify $\mathfrak{gl}_n \simeq \mathfrak{g}_n^*$ using the trace pairing, which is the same way that we identify $\mathrm{Hom}_\mathbb{C}(V, W)$ with $\mathrm{Hom}_\mathbb{C}(W, V)^*$.

Proof sketch. It turns out that in this more general scenario (see [KJ16, Example 9.45]) where our space is symplectic vector space V with symplectic form ω, and a linear action of a group G on V such that ω is G-invariant, then we already have a canonical choice of moment map given by

$$\langle \mu(x), a \rangle = \frac{1}{2}\omega(x, a.x), \quad a \in \mathfrak{g}, \quad x \in V.$$

It suffices to check that this moment map agrees with the one defined above. We directly compute that

$$\begin{aligned}
\langle \mu_\mathbf{v}(z), a \rangle &= \frac{1}{2}\omega(z, a.z), \\
&= \frac{1}{2}\sum_{h \in \Omega}\Big(\mathrm{tr}(z_{\overline{h}}[a, z_h]) - \mathrm{tr}(z_h[a, z_{\overline{h}}])\Big), \\
&= \frac{1}{2}\sum_{h \in \Omega}\Big(\mathrm{tr}(a[z_h, z_{\overline{h}}]) - \mathrm{tr}(a[z_{\overline{h}}, z_h])\Big), \\
&= \sum_{h \in \Omega}\mathrm{tr}(a[z_h, z_{\overline{h}}]).
\end{aligned}$$

Now we identify $\mathfrak{gl}_n \simeq \mathfrak{gl}_n^*$ via the trace pairing and we obtain that

$$\mu_\mathbf{v}(z) = \sum_{h \in \Omega}[z_h, z_{\overline{h}}].$$

□

Remark 4.7: More carefully, we would consider for each $h : i \to j \in \Omega$, the element $z_h z_{\overline{h}}$ to live in $\mathfrak{gl}_{\mathbf{v}_j} \simeq \mathrm{End}(\mathbb{C}^{\mathbf{v}_j})$, and zero in all other $\mathfrak{gl}_{\mathbf{v}_k}$. The summation notation is then reasonably interpreted.

4.3.1 Preprojective algebras through the moment map

This moment map gives us another way to interpret the preprojective algebra. Recalling that Π_λ is a quotient of the path algebra of the double quiver $Q^\#$, we can realize

$$\mathsf{Rep}(\Pi_\lambda, \mathbf{v}) \subset \mathsf{Rep}(Q^\#, \mathbf{v})$$

as the representations satisfying the equation in the ideal we quotient by. But this ideal is nothing else but the moment map! In particular:

$$\mathsf{Rep}(\Pi_\lambda, \mathbf{v}) = \mu_{\mathbf{v}}^{-1}(\lambda).$$

This gives an algebraic meaning to $\mu^{-1}(\lambda)$.

Remark 4.8: Many of the results in the following sections can be rephrased or expanded upon in terms of preprojective algebras, but I'll tend to stay away from them, since it's just another perspective in viewing the preimages of μ. Ultimately we really want to see how these varieties interact with other important objects, for example in representation theory, so I'll be less thorough in explaining each perspective of the intrinsic nature of the varieties.

4.4 GIT moduli spaces for double quivers

Recall that $\mathsf{GL}_{\mathbf{v}}$-orbits in $\mu_{\mathbf{v}}^{-1}(0)$ are in bijection with isomorphism classes of $\mathbf{v}$-dimensional representations of the preprojective algebra Π. This is the double-quiver analog of the fact that $\mathsf{GL}_{\mathbf{v}}$-orbits in $\mathcal{R}(\overrightarrow{Q}, \mathbf{v})$ are in bijection with $\mathbf{v}$-dimensional representations of $\overrightarrow{Q}$, see §3.1. Therefore, we should study the GIT quotient

$$\mu_{\mathbf{v}}^{-1}(0) /\!\!/ \mathsf{GL}_{\mathbf{v}}.$$

Remark 4.9: Once again the action of $\mathsf{GL}_{\mathbf{v}}$ is by conjugation, so the scalar matrices act trivially, hence the action factors through $\mathsf{PGL}_{\mathbf{v}}$.

Now similarly to before, we define the GIT quotients and the locus of stable points.

Definition 4.10: Let $\mathbf{v} \in \mathbb{Z}_{\geq 0}^I$. Define

$$\mathcal{M}_0(\mathbf{v}) := \mu_{\mathbf{v}}^{-1}(0) /\!\!/ \mathsf{PGL}_{\mathbf{v}}$$

to be the GIT quotient. More generally, for $\theta \in \mathbb{Z}^I$ such that $\theta \cdot \mathbf{v} = 0$, define the twisted GIT quotient

$$\mathcal{M}_\theta(\mathbf{v}) := \mu_{\mathbf{v}}^{-1}(0) /\!\!/_{\chi_\theta} \mathsf{PGL}_{\mathbf{v}},$$

and define the subvariety of stable points to be

$$\mathcal{M}_\theta^s(\mathbf{v}) := \{z \in \mathcal{R}(Q^\#, \mathbf{v}) \mid \mu_{\mathbf{v}}(z) = 0, \quad z \text{ is } \chi_\theta\text{-stable}\}/\mathsf{PGL}_{\mathbf{v}} \subset \mathcal{M}_\theta(\mathbf{v}).$$

By the general theory we developed in §3, we see that $\mathcal{M}_0(\mathbf{v})$ is an affine variety, $\mathcal{M}_\theta(\mathbf{v})$ is quasiprojective, and we have a projective morphism

$$\pi : \mathcal{M}_\theta(\mathbf{v}) \to \mathcal{M}_0(\mathbf{v}).$$

Much of the theory carries over basically identically, and the proofs end up being identical as well; therefore, we'll just list these out in a very long theorem.

Theorem 4.11:

(i) $\mathcal{M}_0(\mathbf{v})$ is the set of isomorphism classes of $\mathbf{v}$-dimensional semisimple representations of the preprojective algebra Π. The map $\pi : \mathcal{M}_\theta \to \mathcal{M}_0$ is given by $[V] \mapsto [V^{ss}]$, where V^{ss} is the *semisimplification* of V.

(ii) $\mathcal{M}^s_\theta(\mathbf{v}) \subset \mathcal{M}_\theta(\mathbf{v})$ is open. If it is nonempty, it is a nonsingular variety of dimension $2 - 2\langle \mathbf{v}, \mathbf{v}\rangle$.

(iii) $\mathcal{M}^s_\theta(\mathbf{v})$ contains $T^*\mathcal{R}(\theta^s(\mathbf{v})$ as an open subset.

(iv) One can define the notion of semistable and stable Π-representations (by viewing a representation of Π as a representation of $Q^\#$) in exactly the same way as before, with respect to some $\theta \in \mathbb{R}^I$ with $\theta(\mathbf{v}) = 0$. Then the category of θ-semistable representations of Π is an abelian category, and simple objects are exactly the θ-stable representations.

(v) Let $\theta \in \mathbb{Z}^I$. Then $x \in \mu_{\mathbf{v}}^{-1}(0)$ is χ_θ-semistable in the GIT sense (respectively, stable) iff the corresponding representation $V(x)$ of Π is θ-semistable (respectively, stable) in the classical sense.

(vi) If $Q^\#$ is the double quiver of a Dynkin graph, then $\mathcal{M}_0(\mathbf{v}) = \{\mathrm{pt}\}$ for any $\mathbf{v}$. In particular, $\mathcal{M}_\theta(\mathbf{v})$ is projective.

Remark 4.12: If we assume that $\theta(\mathbf{dim}\, V) = 0$, then the condition for being semistable reduces to the simpler condition that $\theta(\mathbf{dim}\, V') \le 0$ for each proper $V' \subset V$ (and $<$ for stable).

Example 4.13: Let $Q = A_1$ be the graph

$$\bullet \;\text{———}\; \bullet$$

and $\mathbf{v} = (1, 1)$. Then

$$Q^\# = \bullet \rightleftarrows \bullet$$

and we identify

$$\mathcal{R}(Q^\#, \mathbf{v}) = \mathbb{C} \oplus \mathbb{C},$$

where the first coordinate is the space of maps $\mathbb{C} \to \mathbb{C}$, and the second coordinate is the space of maps $\mathbb{C} \leftarrow \mathbb{C}$. Letting $(x, y) \in \mathcal{R}(Q^\#, \mathbf{v})$, we see that

$$(x, y) \in \mu_{\mathbf{v}}^{-1}(0) \iff xy = 0 = yx \implies \mu_{\mathbf{v}}^{-1}(0) \simeq \operatorname{Spec} \mathbb{C}[x, y]/(xy).$$

Now $\mathcal{M}_0(\mathbf{v})$ is the GIT quotient of this; the quotient by the $\mathrm{GL}_{\mathbf{v}}$-action gives us just the fixed points under this action, which is $(0, 0)$, hence

$$\mathcal{M}_0(\mathbf{v}) = \{\mathrm{pt}\}.$$

4.5 McKay correspondence

A really amazing example comes from McKay correspondence.

To a finite subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$, there is an associated extended simply-laced Dynkin diagram Q_Γ, which we can realize as a double quiver $Q_\Gamma^\#$ by doubling the arrows. (Therefore Q_Γ is of type $\widehat{A}$, $\widehat{D}$, or $\widehat{E}$.) In fact the McKay correspondence says that there is a bijection of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ up to isomorphism (or even just conjugacy), and simply-laced Dynkin diagrams.

Since $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ it acts on $\mathbb{C}^2$; pick a basis x, y. Then we can concretely understand the preprojective algebra $\Pi_0(Q_\Gamma)$.

Theorem 4.14:

(a) There is an algebra isomorphism $\Pi_0(Q_\Gamma) \simeq e\,[\mathbb{C}[x, y] \rtimes \Gamma]\,e$ for a specific idempotent $e \in \mathbb{C}[\Gamma]$. In particular, the algebras $\Pi_0(Q_\Gamma)$ and $\mathbb{C}[x, y] \rtimes \Gamma$ are Morita equivalent.

(b) There is a canonical algebra isomorphism $\widetilde{e}\Pi_0(Q_\Gamma)\widetilde{e} \simeq \mathbb{C}[x, y]^\Gamma$ for the idempotent $\widetilde{e} = \sum_{i \in I} e_i$.

To an extended Dynkin diagram, there is a unique minimal imaginary root of the associated root system (which is the root system of a simple affine Kac-Moody Lie algebra); call this δ. This is now concretely realized as an element in $\mathbb{Z}^I$. The $\mathcal{M}_\theta(\delta)$ are very interesting varieties indeed.

Theorem 4.15:

(a) $\mathcal{M}_0(\delta) \simeq \mathbb{C}^2/\Gamma \coloneqq \operatorname{Spec} \mathbb{C}[x,y]^\Gamma$.

(b) If $\theta \in \mathbb{Z}^I$ does not live in any root hyperplane of the affine root system, then $\mathcal{M}_\theta(\delta)$ is smooth, and the canonical morphism

$$\mathcal{M}_\theta(\delta) \to \mathcal{M}_0(\delta)$$

is a minimal resolution of singularities.

So we can concretely realize the geometric McKay correspondence via quiver moduli!

4.6 Framing

Framings will fix the shortcomings of the moduli spaces of stable representations, which are usually empty.

First let's discuss framing in the case of ordinary quivers (not double quivers). Let $\overrightarrow{Q}$ be a quiver with vertices I and (oriented) edges Ω.

Definition 4.16 (framing): Let W be an I-tuple of vector spaces $(W_i)_{i\in I}$. A W-**framed representation** of $\overrightarrow{Q}$, is a representation $V = (V_i, x_h)$ (for $i \in I$ and $h \in \Omega$) of $\overrightarrow{Q}$, together with the data of linear maps

$$j_i : V_i \to W_i$$

for each $i \in I$.

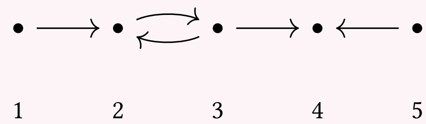
A **morphism** $f : V \to V'$ of two W-framed representations (with the same W) is a morphism of representations of $\overrightarrow{Q}$, which commutes with the j_i maps:

$$j_i' \circ f_i = j_i : \quad V_i \to W_i.$$

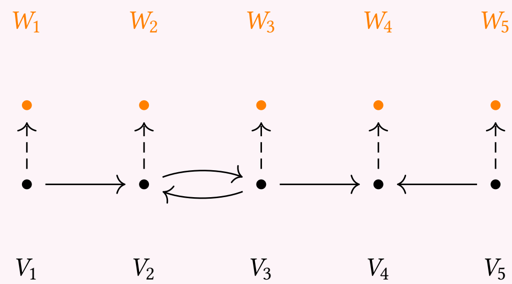
Remark 4.17: There is a slightly different, but equivalent, formulation of framing in [CB01].

Remark 4.18: Framed representations are closely related with representations of a new quiver, obtained from $\overrightarrow{Q}$ by adding, for every $i \in I$, a new vertex $\widehat{i}$ and an edge $i \to \widehat{i}$. The difference is that morphisms of framed representations are required to be identity on W, while morphisms of quivers can be anything on each vertex. As a result, **morphisms of framed representations do not form an abelian group**.

Example 4.19: Consider the quiver



with the edges labeled 1 through 5. Here is an example of a framed representation:



Here, the V_i are a representation of the above quiver, and the W_i form a framing W of the representation V.

Remark 4.20: Perhaps this is why it's called "framing." It's like the V_i are a picture and the W_i are a frame to hang it up on the wall. I also totally made this up just now.

We will see why framing is important later. For now, let us review basic properties. First note that W is determined uniquely by $\mathbf{dim}\, W$, up to isomorphism (as we do not have maps between the W_i). For a given $\mathbf{v} \in \mathbb{Z}^I_{\geq 0}$, we can form a W-framed representation of graded dimension $\mathbf{v}$ as follows. First we can choose any representation V of graded dimension $\mathbf{v}$. Then for each vertex $i \in I$ we can choose any map $V_i \simeq \mathbb{C}^{\mathbf{v}_i} \longrightarrow W_i$. Therefore let us define

$$R(V, W) \coloneqq \left(\bigoplus_{i \in I} \mathrm{Hom}_{\mathbb{C}}(V_i, W_i)\right) \oplus \underbrace{\left(\bigoplus_{e: i \to j \in \Omega} \mathrm{Hom}_{\mathbb{C}}(V_i, V_j)\right)}_{= \mathcal{R}(\overrightarrow{Q}, V)},$$

and for $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^I_{\geq 0}$ also define

$$R(\mathbf{v}, \mathbf{w}) \coloneqq R(\mathbb{C}^{\mathbf{v}}, \mathbb{C}^{\mathbf{w}}).$$

It is clear that $R(V, W) \simeq R(\mathbf{dim}\, V, \mathbf{dim}\, W)$. Of course, the point is that there is a natural bijection

$$\{\mathrm{GL}_{\mathbf{v}}\text{-orbits in } R(\mathbf{v}, \mathbf{w})\} \leftrightarrow \{\text{isomorphism classes of } \mathbb{C}^{\mathbf{w}}\text{-framed representations of } \overrightarrow{Q} \text{ of dimension } \mathbf{v}\}.$$

However, once again, the quotient space is not Hausdorff so we will have to make do with the GIT quotient

$$\mathcal{R}_0(\mathbf{v}, \mathbf{w}) \coloneqq R(\mathbf{v}, \mathbf{w}) /\!\!/ \mathrm{GL}_{\mathbf{v}}.$$

Remark 4.21: Note that this time the action does **not** factor through $\mathrm{PGL}_{\mathbf{v}}$! This is because $\mathrm{GL}_{\mathbf{v}}$ acts by conjugation only on the edges in V; it actually acts on each vertex. So while scalars do act trivially on the edges in V (by scaling up the head but scaling down the tail by the same amount), they **don't** act trivially on the framed edges $j_i : V_i \longrightarrow W_i$, as they only scale up V_i (but not W_i).

Theorem 4.22: For any quiver $\overrightarrow{Q}$, we have

$$\mathcal{R}_0(\mathbf{v}, \mathbf{w}) \simeq \mathcal{R}_0(\mathbf{v})$$

is independent of parameter $\mathbf{w}$.

So framing does not give us any new moduli spaces.

Proof. Once again, $\mathcal{R}_0(\mathbf{v},\mathbf{w})$ is the set of closed orbits. So it suffices to prove that if $(x,j) \in R(\mathbf{v},\mathbf{w})$ lives in a closed orbit, then $j_i = 0$ for all $i \in I$. But closed orbits contain limit points of the action of one-parameter subgroups in $\mathsf{GL}_\mathbf{v}$; consider the one-parameter subgroup $\lambda(t) = t^{-1} \cdot \mathrm{Id}$. Then each $\lambda(t)$ acts trivially on all x (morphisms in V), but scales each $j_i : V_i \to W_i$ by $\lambda(t) \cdot (x,j) = (x, tj)$. Taking the limit as $t \to 0$ gives $(x, 0)$, which must be in this orbit (since it's closed). Therefore each closed orbit has a point where $j = 0$, and so the quotient is nothing new. □

Corollary 4.23: In particular, if $\overrightarrow{Q}$ is Dynkin, then $\mathcal{R}_0(\mathbf{v},\mathbf{w})$ is a point.

We can also consider twisted GIT quotients. Let $\theta \in \mathbb{Z}^I$, $\chi_\theta : \mathsf{GL}_\mathbf{v} \to \mathbb{C}^\times$ the corresponding character (see §3.6). In fact we are mostly interested in two special cases:

- all $\theta_i > 0$, in which case we write $\theta > 0$;
- all $\theta_i < 0$, in which case we write $\theta < 0$.

Theorem 4.24: Let $\overrightarrow{Q}$ be an arbitrary quiver. Let $\theta \in \mathbb{Z}^I$, $\theta > 0$ in the sense just above, with corresponding character χ_θ. Then:
(a) Point $(x,j) \in R(\mathbf{v},\mathbf{w})$ is χ_θ-semistable (in the GIT sense) iff every $\overrightarrow{Q}$-subrepresentation of $\ker(j)$ of V_x is zero.
(b) Any χ_θ-semistable point is automatically stable.

The proof is very similar to Theorem 3.18 but with minor changes, so we omit it, but you can find it in [KJ16, Theorem 10.22].

Corollary 4.25: Note that the second condition in part (a) is independent of θ. In particular any two positive stability parameters $\theta, \theta' > 0$ define the same stability condition:

$$\theta, \theta' > 0 \implies \mathcal{R}_\theta(\mathbf{v},\mathbf{w}) = \mathcal{R}_{\theta'}(\mathbf{v},\mathbf{w}).$$

Corollary 4.26: Let $\theta > 0$ such that $\mathcal{R}_\theta(\mathbf{v},\mathbf{w}) \neq \emptyset$. Then $\mathcal{R}_\theta(\mathbf{v},\mathbf{w})$ is a smooth variety of dimension

$$\mathbf{v} \cdot \mathbf{w} - \langle \mathbf{v}, \mathbf{v} \rangle.$$

Proof. By Theorem 4.24 we have

$$\mathcal{R}_\theta(\mathbf{v},\mathbf{w}) = \mathcal{R}_\theta(\mathbf{v},\mathbf{w})^{ss}_{\chi_\theta} /\!\!/_\chi \mathsf{GL}_\mathbf{v} = \mathcal{R}_\theta(\mathbf{v},\mathbf{w})^{s}_{\chi_\theta} / \mathsf{GL}_\mathbf{v}.$$

But then Theorem 3.13(c) tells us that the dimension can be computed by $\dim R(\mathbf{v},\mathbf{w}) - \dim \mathsf{GL}_\mathbf{v}$, from which we get $\mathbf{v} \cdot \mathbf{w} - \langle \mathbf{v}, \mathbf{v} \rangle$. □

Once again, we can describe the map

$$\pi : \mathcal{R}_\theta(\mathbf{v},\mathbf{w}) \to \mathcal{R}_0(\mathbf{v},\mathbf{w}).$$

It is given by

$$\pi([x,j]) = [x^{ss}, 0],$$

where x^{ss} is the semisimplification of the representation V_x. Compare this to Remark 3.12!

Example 4.27: Let $\theta > 0$ and $\overrightarrow{Q} = \bullet$. Then representations of $\overrightarrow{Q}$ are just vector spaces, indexed by nonnegative integers. Theorem 4.24(a) tells us that

$$\mathcal{R}_\theta(a,b) = \{ j : \mathbb{C}^n \to \mathbb{C}^r \mid \ker(j) = 0 \} / \mathsf{GL}_n = \mathsf{Gr}(n,r),$$

the Grassmannian of n-dimensional subspaces in $\mathbb{C}^r$.

Example 4.28: Let $\theta > 0$ and $\overrightarrow{Q}$ be a quiver of type A_k:

$$\overrightarrow{Q} = \bullet \to \bullet \to \cdots \to \bullet \to \bullet.$$

Let $\mathbf{w} = (0, 0, \ldots, 0, r)$, i.e. a single $W_k = \mathbb{C}^r$ above the last vertex. Then a point $(x_1, \ldots, x_{k-1}, j) \in R(\mathbf{v}, \mathbf{w})$ is semistable iff every $\overrightarrow{Q}$-subrepresentation of $\ker(j)$ is zero. But $\ker(j)$ is exactly

$$\begin{array}{ccccccccc} & & & & & & & & \mathbb{C}^r \\ & & & & & & & & \uparrow j_k \\ V_1 & \xrightarrow{x_1} & V_2 & \xrightarrow{x_2} & \cdots & \xrightarrow{x_{k-2}} & V_{k-1} & \xrightarrow{x_{k-1}?} & \ker(j_k) \subset V_k \end{array}.$$

So we need to know that these all vanish. First obvious j_k needs to be injective so that $\ker(j_k) = 0$. Then $\ker(x_{k-1})$ must be zero, i.e. injective, as any kernel defines a subrepresentation. Continuing in this way we see that each $V_i \overset{x_i}{\hookrightarrow} V_{i+1}$ must be an injection, so in total this defines a flag

$$V_1 \hookrightarrow V_2 \hookrightarrow \cdots \hookrightarrow V_k \hookrightarrow \mathbb{C}^r$$

with $\dim V_i = \mathbf{v}_i$. Now quotienting by the $\mathrm{GL}_{\mathbf{v}}$-action, we get that

$$\mathcal{R}_\theta(\mathbf{v}, \mathbf{w}) = \mathcal{F}\ell(\mathbf{v}_1, \ldots, \mathbf{v}_k, r)$$

the partial flag variety of flags of dimension $\mathbf{v}_i$ inside $\mathbb{C}^r$. (Therefore we see that in order for such a point to possibly be semistable, we must have $\mathbf{v}_1 \le \mathbf{v}_2 \le \cdots \le \mathbf{v}_k \le r$.)

4.7 Framed representations of double quivers

Now we get to what we really care about: framed representations of *double* quivers. Let $Q^\#$ be a double quiver with vertices I and $H = \Omega \cup \overline{\Omega}$ a choice of decomposition of its edges, i.e. $Q^\#$ is the double quiver of a quiver $\overrightarrow{Q}$ with oriented edges Ω.

Definition 4.29 (framing of double quiver): Let W be an I-tuple of vector spaces. A W**-framed representation of** $Q^\#$ is a representation $V = (V_i, x_h)$ (for $i \in I$ and $h \in H$) of $Q^\#$, together with the data of linear maps

$$j_i : V_i \to W_i, \quad \bar{j}_i : W_i \to V_i$$

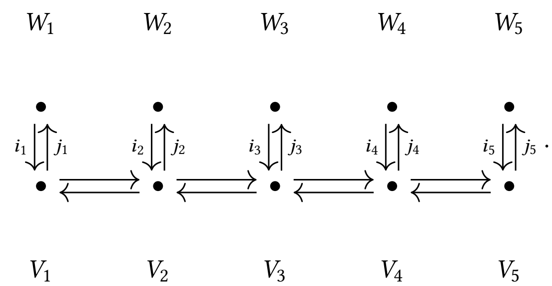
for each i.

In other words a framing of the double quiver now requires us to have maps going **both ways** from the "framed" part.

The obvious analogue of the previous $R(V, W)$ is to define

$$R(Q^\#, V, W) \coloneqq \underbrace{\left(\bigoplus_{e:i\to j\in\Omega} \mathrm{Hom}_{\mathbb{C}}(V_i, V_j)\right)}_{=\mathcal{R}(Q^\#, V)} \oplus \underbrace{\left(\bigoplus_{i\in I} \mathrm{Hom}_{\mathbb{C}}(V_i, W_i)\right)}_{=\text{maps to the frame}} \oplus \underbrace{\left(\bigoplus_{i\in I} \mathrm{Hom}_{\mathbb{C}}(W_i, V_i)\right)}_{=\text{maps from the frame}}.$$

We will write elements of $R(Q^\#, V, W)$ as triples $m = (z, j, i)$ for $z \in \mathcal{R}(Q^\#, V)$, $j : V \to W$, and $i : W \to V$, as shown below:



Remark 4.30: $Q^\#$ inherently doesn't remember any orientation, and as such $R(Q^\#, V, W)$ is defined independently of the choice of orientation $H = \Omega \cup \overline{\Omega}$. However for each choice of such orientation, we get a canonical isomorphism

$$R(Q^\#, V, W) \simeq T^* R(\overrightarrow{Q}, V, W).$$

In fact more generally for every skew symmetric function $\varepsilon : H \longrightarrow \mathbb{C}^\times$ we get a symplectic form on $R(Q^\#, V, W)$.

Of course now we do the usual thing: quotient by $\mathrm{GL}_{\mathbf{V}}$-action. In exactly the same fashion as in Theorem 4.5, we get

Theorem 4.31: The action of $\mathrm{GL}_{\mathbf{v}}$ on $R(Q^\#, V, W)$ is Hamiltonian, and the moment map

$$\mu_{V,W} : R(Q^\#, V, W) \longrightarrow \mathfrak{gl}(V)$$

is given by

$$\mu_{V,W}(z, j, i) = \sum_{h \in \Omega} [z_h, z_{\overline{h}}] - \sum_{k \in I} i_k j_k.$$

Remark 4.32: Some fairly trivial comments:

- Of course we can replace V, W with their dimensions $\mathbf{v}, \mathbf{w}$.
- $\mathfrak{gl}(V) \coloneqq \bigoplus_{k \in I} \mathfrak{gl}(V_i)$.
- We again identify $\mathfrak{gl}(V) \simeq \mathfrak{gl}(V)^*$ using the trace pairing.
- The proof is essentially just combining the arguments from before, so it's omitted.

As a result, we can now consider the corresponding GIT quotients.

4.8 Nakajima quiver variety

At this point we'll drop the Q from the notation for simplicity.

Definition 4.33: For $\theta \in \mathbb{Z}^I$ and $\lambda = (\lambda_i \operatorname{Id}_{V_i})_{i \in I} \in \mathbb{C}^I$, define the **Nakajima quiver variety**

$$\mathcal{M}_{\theta,\lambda}(\mathbf{v}, \mathbf{w}) \coloneqq \mu_{\mathbf{v},\mathbf{w}}^{-1}(\lambda) /\!\!/_{\chi_\theta} \mathrm{GL}_{\mathbf{v}} = \mu^{-1}(\lambda)_\theta^{ss} /_{\chi_\theta} \mathrm{GL}_{\mathbf{v}}.$$

There's also a subset of regular elements:

Definition 4.34: Let $\mu_{\mathbf{v},\mathbf{w}}^{-1}(\lambda)^{\mathrm{reg}} \subset \mu_{\mathbf{v},\mathbf{w}}^{-1}(\lambda)$ denote the subset of points with trivial stabilizer (i.e. isotropy group) and whose orbits is closed. Then for $\theta = 0$, define

$$\mathcal{M}_{0,\lambda}(\mathbf{v}, \mathbf{w})^{\mathrm{reg}} \subset \mathcal{M}_{0,\lambda}(\mathbf{v}, \mathbf{w})$$

to be the image of this set inside $\mu_{\mathbf{v},\mathbf{w}}^{-1} /\!\!/ \mathrm{GL}_{\mathbf{v}}$.

Nakajima proved that $\mathcal{M}_{0,\lambda}(\mathbf{v}, \mathbf{w})^{\mathrm{reg}} \subset \mathcal{M}_{0,\lambda}(\mathbf{v}, \mathbf{w})$ is always open, although sometimes empty.

Proposition 4.35: Suppose $\overrightarrow{Q}$ has no edge loops and (θ, λ) is a "$\mathbf{v}$-regular parameter" (a technical condition; see [Gin09, page 22]; if $\lambda = 0$ and $\theta > 0$ then this condition is always satisfied). Then:

(a) Any point in $\mu_{\mathbf{v},\mathbf{w}}^{-1}(\lambda)^{\text{reg}}$ is θ-stable.

(b) If $\mathcal{M}_{0,\lambda}(\mathbf{v}, \mathbf{w})^{\text{reg}}$ is nonempty, then it is dense inside $\mathcal{M}_{0,\lambda}(\mathbf{v}, \mathbf{w})$, and furthermore:

- the canonical projective morphism $\pi : \mathcal{M}_{\theta,\lambda}(\mathbf{v}, \mathbf{w}) \to \mathcal{M}_{0,\lambda}(\mathbf{v}, \mathbf{w})$ is a symplectic resolution of singularities,
- the subvariety $\pi^{-1}(\mathcal{M}_{0,\lambda}(\mathbf{v}, \mathbf{w})^{\text{reg}})$ is dense inside $\mathcal{M}_{\theta,\lambda}(\mathbf{v}, \mathbf{w})$,
- the map π restricts to an isomorphism

$$\pi : \quad \pi^{-1}(\mathcal{M}_{0,\lambda}(\mathbf{v}, \mathbf{w})^{\text{reg}}) \xrightarrow{\sim} \mathcal{M}_{0,\lambda}(\mathbf{v}, \mathbf{w})^{\text{reg}}.$$

Remark 4.36: There is a combinatorial criterion for $\mathcal{M}_{0,\lambda}(\mathbf{v}, \mathbf{w})^{\text{reg}}$ to be nonempty.

Remark 4.37: As twisted GIT quotients, Nakajima quiver varieties parametrize equivalence classes of θ-semistable *something*. In this case, that something can be realized as Π_λ-modules (where Π_λ is the deformed preprojective algebra), and the equivalence $x \sim x'$ is if the closures of the orbits of x, x' intersect in the χ_θ-semistable locus; compare this to Definition 3.6, where we defined (semi)stability of points. (This is what [Gin09] calls "S-equivalence."

Remark 4.38: There is another way to construct the GIT quotients in both of the cases (of quivers and double quivers) using hyperkähler quotients. In particular we can construct the Nakajima quiver varieties as hyperkähler quotients, and they can be equipped with the structure of a hyperkähler manifold.

5 Properties and applications of Nakajima quiver varieties

Much of our studies will actually concern the case $\lambda = 0$, so sometimes we will omit the notation from $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w}) \coloneqq \mathcal{M}_{\theta,0}(\mathbf{v}, \mathbf{w})$.

5.1 Stability conditions

I'm not particularly interested in spelling out the details of stability conditions here, so the interested reader should consult [KJ16, §10.5] or [Gin09]. I'll just remark that stability conditions for $R(Q^\#, \mathbf{v}, \mathbf{w})$ (in the GIT sense) once again can be described in terms of quiver representations, just as before: we can define what it means for a framed representation of $Q^\#$ to be (semi)stable with respect to some parameter θ. In fact for some notion of θ being "generic," we can even say that θ-semistable implies θ-stable. For completeness, we'll spell out the definition.

Definition 5.1 (v-generic): Define

$$R_+(\mathbf{v}) \coloneqq \{\alpha \in \mathbb{Z}_{\geq 0}^I \mid \alpha \neq 0,\ (\alpha, \alpha) \leq 2,\ \mathbf{v} - \alpha \in \mathbb{Z}_{\geq 0}\}.$$

(In the case of Dynkin or extended Dynkin graphs, this is exactly the positive roots "smaller" than $\mathbf{v}$.)

Then $\theta \in \mathbb{R}^I$ is called **v-generic** if for any $\alpha \in R_+(\mathbf{v})$, $\theta \cdot \alpha \neq 0$.

Remark 5.2: Being $\mathbf{v}$-generic can be rephrased as being in the complement of the hyperplanes defined by $R_+(\mathbf{v})$.

The main result:

Theorem 5.3: Let $\theta \in \mathbb{Z}^I$ be $\mathbf{v}$-generic. Assume that $\mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w}) \neq \emptyset$. Then:

(a) $\mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w})$ is nonsingular, of dimension

$$\dim \mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w}) = 2\mathbf{v}\cdot\mathbf{w} - 2\langle\mathbf{v},\mathbf{v}\rangle.$$

(b) The Poisson structure on $\mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w})$ is nondegenerate, i.e., it comes from a symplectic form.

(c) Pick an orientation so that $H = \Omega \cup \overline{\Omega}$, so that $Q^\#$ is the double quiver of $\overrightarrow{Q}$. Then

$$R(Q^\#,\mathbf{v},\mathbf{w}) = T^*R(\mathbf{v},\mathbf{w}).$$

If $\theta > 0$, then $\mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w})$ contains $T^*\mathcal{R}_\theta(\mathbf{v},\mathbf{w})$ as an open, but possibly empty, subset.

(d) The variety $\mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w})$ is connected.

5.2 Symplectic resolutions

Recall that we have a canonical projective morphism

$$\pi : \mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w}) \to \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w}).$$

This map turns out to be extremely interesting.

Theorem 5.4: Let $\theta \in \mathbb{Z}^I$ be $\mathbf{v}$-generic, and assume that $\mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})^{\mathrm{reg}} \neq \emptyset$. Then

$$\pi : \mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w}) \to \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$$

is a symplectic resolution of singularities.

In particular, $\mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w})$ is smooth and irreducible.

Remark 5.5: A point in $R(Q^\#, V, W)$ being regular can be concretely described in terms of the corresponding framed representation of the double quiver, see [KJ16, Theorem 10.41].

5.3 $\mathbb{C}^\times$-action and the exceptional fiber

In this subsection we assume Q is a graph without edge loops and $\overrightarrow{Q}$ is the induced quiver from choosing an orientation Ω of its edges, so that

$$R(Q^\#,\mathbf{v},\mathbf{w}) \simeq T^*R(\overrightarrow{Q},\mathbf{v},\mathbf{w}).$$

Write elements of $R(Q^\#,\mathbf{v},\mathbf{w})$ as (x,y,i,j) where $x \in \mathcal{R}(\overrightarrow{Q},\mathbf{v})$ and $y \in \mathcal{R}(\overrightarrow{Q}^{op},\mathbf{v})$. We will define a $\mathbb{C}^\times$-action on $R(Q^\#,\mathbf{v},\mathbf{w})$ by

$$t\cdot(x,y,i,j) = (x,ty,ti,j), \quad t \in \mathbb{C}^\times.$$

It is clear that this $\mathbb{C}^\times$-action commutes with the $\mathrm{GL}_\mathbf{v}$-action, and the symplectic form and moment map are both homogeneous degree 1 with respect to the $\mathbb{C}^\times$-action. As a result, for any $\theta \in \mathbb{Z}^I$, the $\mathbb{C}^\times$-action descends to a $\mathbb{C}^\times$-action on $\mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w})$, and the projective morphism $\pi : \mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w}) \to \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$ is $\mathbb{C}^\times$-equivariant.

We will now apply the approach of Bialynicki-Birula to this $\mathbb{C}^\times$-action; see [CG97, §2.4]. To start, we need to know what the fixed points are.

Proposition 5.6: For each $x \in \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$, then

$$\lim_{t\to 0} t\cdot x = 0,$$

and the only fixed point of the $\mathbb{C}^\times$-action on $\mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$ is the point 0. (Furthermore, the only point whose $\lim_{t\to\infty} t\cdot x$ exists is $x = 0$.)

Proof sketch. It's straightforward to see that $\lim_{t \to 0} t \cdot (x, y, i, j) = (x, 0, 0, j) \in R(Q^\#, \mathbf{v}, \mathbf{w})$, and the closure of this orbit must contain the point $(x, 0, 0, 0)$. Since $\overrightarrow{Q}$ has no oriented cycles, this closure also contains $(0, 0, 0, 0)$. □

Let us now fix $\theta \in \mathbb{Z}^I$ to be $\mathbf{v}$-generic, so that $\mathcal{M}_{\theta,0}(\mathbf{v}, \mathbf{w})$ is smooth (see Theorem 5.4). Let

$$F \coloneqq \mathcal{M}_{\theta,0}(\mathbf{v}, \mathbf{w})^{\mathbb{C}^\times}$$

be the $\mathbb{C}^\times$-fixed points; since $\mathcal{M}_{\theta,0}(\mathbf{v}, \mathbf{w})$ is smooth and $\mathbb{C}^\times$ is reductive, then F is also smooth. The $\mathbb{C}^\times$-equivariance of π immediately implies that

$$F \subset \pi^{-1}(0).$$

Let

$$F = \bigsqcup F_s$$

denote the decomposition of F into connected components.

Definition 5.7 (exceptional fiber): Assume that $\theta \in \mathbb{Z}^I$ is $\mathbf{v}$-generic. We define the **exceptional fiber**

$$\mathcal{L}_\theta(\mathbf{v}, \mathbf{w}) \coloneqq \pi^{-1}(0) \subset \mathcal{M}_{\theta,0}(\mathbf{v}, \mathbf{w}).$$

Then by the general results of Bialynicki-Birula we get:

Proposition 5.8: For $\theta \in \mathbb{Z}^I$ $\mathbf{v}$-generic, then

(a) $\mathcal{L}_\theta(\mathbf{v}, \mathbf{w}) = \bigcup \mathcal{L}_s$, where

$$\mathcal{L}_s \coloneqq \left\{ m \in \mathcal{M}_\theta(\mathbf{v}, \mathbf{w}) \mid \lim_{t \to \infty} t \cdot m \text{ exists and is in } F_s \right\}.$$

(b) Each $\mathcal{L}_s$ is a smooth, connected, locally closed Lagrangian subvariety in $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w})$.

The $\mathcal{L}_s$ are called the **Bialynicki-Birula pieces**.

Corollary 5.9: In particular, the irreducible components of $\mathcal{L}_\theta(\mathbf{v}, \mathbf{w})$ are exactly the Zariski closures of $\mathcal{L}_s$.

Remark 5.10: This proposition can be false for quivers with oriented loops, e.g. the Jordan quiver.

Remark 5.11: For $\theta > 0$, we can explicitly describe $\mathcal{L}_\theta(\mathbf{v}, \mathbf{w})$. Let $m = (x, y, i, j) \in \mu_{\mathbf{v},\mathbf{w}}^{-1}(0)$ be θ-semistable. Then $[m] \in \mathcal{L}_\theta(\mathbf{v}, \mathbf{w})$ iff $i = 0$ and $z = (x, y)$ is nilpotent as a $Q^\#$-representation (recall that this means that there exists N such that for any path in $Q^\#$ of length $\geq N$, then the composition of those maps in z are zero, i.e. $z_{h_k} \cdots z_{h_1} = 0$ for all paths $h_k, \ldots, h_1$ of length $k \geq N$).

In fact one can define a variety $\Lambda(\mathbf{v})$ of nilpotent representations of $Q^\#$, see [KJ16, §5.3]. Comparing the above description, we have

$$\mathcal{L}_\theta(\mathbf{v}, \mathbf{w}) = (\Lambda(\mathbf{v}) \times \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\mathbf{v}}, \mathbb{C}^{\mathbf{w}})) /\!\!/_{\chi_\theta} \mathrm{GL}_{\mathbf{v}}.$$

Using this description we find that we have a natural injective map

$$\mathrm{Irr}(\mathcal{L}_\theta(\mathbf{v}, \mathbf{w})) \hookrightarrow \mathrm{Irr}(\Lambda(\mathbf{v})),$$

where Irr is the set of irreducible components.

Lastly, we conclude with a nice topological fact about the exceptional fiber.

Proposition 5.12: Let θ be $\mathbf{v}$-generic. Then the inclusion $\mathcal{L}_\theta(\mathbf{v}, \mathbf{w}) \hookrightarrow \mathcal{M}_\theta(\mathbf{v}, \mathbf{w})$ is a homotopy equivalence.

5.4 Hilbert schemes

This subsection is based on [KJ16, §11].

The goal of this subsection is to study Nakajima quiver varieties for the Jordan quiver

$$\overrightarrow{Q} \coloneqq$$

.

Amazingly, the varieties that come up are Hilbert schemes! Let's briefly review what they are.

5.4.1 Hilbert schemes

Suppose we have X an affine algebraic variety over $\mathbb{C}$. If we want to parametrize n points in X, we might try to construct the symmetric power of X:

$$S^n X \coloneqq X^n /\!\!/ S_n = \operatorname{Spec}\left(\mathbb{C}[X]^{\otimes n}\right)^{S_n}.$$

Example 5.13: Let $X = \mathbb{A}^1$. Then

$$S^n \mathbb{A}^1 = \operatorname{Spec}\left(\mathbb{C}[x_1, \dots, x_n]^{S_n}\right) \simeq \operatorname{Spec} \mathbb{C}[s_1, \dots, s_n] \simeq \mathbb{A}^n,$$

by a classical fact about symmetric polynomials.

In general, however, $S^n X$ is singular - even when X is smooth. Actually there is a better variety to work with (although it can still be singular), called the **Hilbert scheme** (of n points).

Definition 5.14: Let $X = \operatorname{Spec} A$ an affine algebraic variety. The **Hilbert scheme of n points** $\operatorname{Hilb}^n X$ is defined as a set to be

$$\{J \subset A \mid \text{ideals such that } \dim A/J = n\}.$$

In other words, it's closed subsets which correspond to "n points," but counting multiplicity of points. It can be shown that $\operatorname{Hilb}^n X$ can be made into a scheme. Here are some important facts about Hilbert schemes (of points).

Theorem 5.15:

(a) There is a canonical projective morphism called the **Hilbert-Chow morphism**

$$\pi : \operatorname{Hilb}^n X \to S^n X, \quad J \mapsto \operatorname{supp}(\mathbb{C}[X]/J)$$

where support is considered as a set of points of X counted with multiplicities.

(b) We have the open subset

$$S_0^n X = \{(t_1, \dots, t_n) \in X^n \mid t_i \neq t_j\}/S_n \subset S^n X.$$

Let $\operatorname{Hilb}_0^n X \coloneqq \pi^{-1}(S_0^n X)$. This is an open subset of $\operatorname{Hilb}^n X$, and the restriction of the Hilbert-Chow morphism to $\operatorname{Hilb}_0^n X$ is an isomorphism.

(c) If X is a nonsingular variety of dimension 2, then:

- $\operatorname{Hilb}_0^n X$ is dense in $\operatorname{Hilb}^n X$.
- $\operatorname{Hilb}^n X$ is smooth.
- The Hilbert-Chow morphism is a resolution of singularities.

5.4.2 The Jordan quiver

Let us return to the Jordan quiver,

$\overrightarrow{Q} \coloneqq$  .

First let us consider the unframed varieties.

Example 5.16: There's only one vector space to work with here, so $\mathbf{v}$ is an integer. Let $\mathbf{v} = n$. Then obviously $\mathcal{R}(\overrightarrow{Q}, n) = \mathrm{End}(\mathbb{C}^n)$, so

$$\mathcal{R}_0(\overrightarrow{Q}, n) = \mathcal{R}(\overrightarrow{Q}, n) /\!\!/ \mathrm{PGL}_n \simeq \mathbb{C}^n / S_n \simeq \mathbb{C}^n.$$

It can be shown that the corresponding variety for the double quiver is

$$\mathcal{M}_0(\overrightarrow{Q}, n) \simeq \mathbb{C}^{2n}/S_n,$$

which is singular.

Now let us consider the *framed* versions. This is where Hilbert schemes show up!

Theorem 5.17: Let $\theta < 0$, $\mathbf{v} = n$, and $\mathbf{w} = 1$. Then:

(a) $\mathcal{M}_0(n, 1) \simeq S^n \mathbb{C}^2$.
(b) $\mathcal{M}_\theta(n, 1) \simeq \mathrm{Hilb}^n \mathbb{C}^2$, and it is smooth symplectic variety.
(c) The canonical morphism $\pi : \mathcal{M}_\theta(n, 1) \to \mathcal{M}_0(n, 1)$ is a Poisson morphism, and coincides with the Hilbert-Chow morphism; they are both symplectic resolutions of singularities.

All of this is just for $\mathcal{M}_\theta(n, 1)$! If we let $\mathbf{w} > 1$, we get other very interesting spaces as well.

5.4.3 Kleinian singularities

One deep and very interesting application of Hilbert schemes is to Kleinian singularities, and of course to the McKay correspondence. I only want to discuss how Nakajima quiver varieties come into play with the McKay correspondence, so I won't explain the McKay correspondence much here; see for example [KJ16] for a detailed explanation.

Let $G \subset \mathrm{SL}(2, \mathbb{C})$ be a finite subgroup, acting naturally on $\mathbb{C}^2$. A **Kleinian singularity** is a singular scheme of the form $\mathbb{C}^2/G$. Now because $G \curvearrowright \mathbb{C}^2$, then $G \curvearrowright \mathrm{Hilb}^n \mathbb{C}^2$, where $n = |G|$.

Definition 5.18: Define $\mathrm{Hilb}^G \mathbb{C}^2$ to be the G-fixed points of $\mathrm{Hilb}^n \mathbb{C}^2$, where $n = |G|$. It is known that this is a nonsingular variety.

In fact $\mathrm{Hilb}^G \mathbb{C}^2$ decomposes into nonsingular subvarieties indexed by certain representations of the McKay quiver. We will focus on one: δ, corresponding to the regular representation of G. Explicitly,

$$X^\delta \coloneqq \{J \in \mathrm{Hilb}^G \mathbb{C}^2 \mid \mathbb{C}[\mathrm{Hilb}^G \mathbb{C}^2]/J \simeq \bigoplus_i (\dim \rho_i)\rho_i \text{ as a representation of } G\},$$

where the ρ_i are the irreducible representations of G. The key facts that will relate this back to what we just looked at are:

Theorem 5.19:
(a) $(S^n\mathbb{C}^2)^G \simeq \mathbb{C}^2/G$ for $n = |G|$.
(b) X^δ is connected, and the Hilbert-Chow morphism induces a map $X^\delta \to \mathbb{C}^2/G$ (which we will still refer to as the Hilbert-Chow morphism π) which is a resolution of singularities.
(c) In fact the Hilbert-Chow morphism $\pi : X^\delta \to \mathbb{C}^2/G$ coincides with the *minimal* resolution of singularities of $\mathbb{C}^2/G$ (e.g., given by blowing up the origin enough times). In particular, the exceptional fiber is a bunch of $\mathbb{P}^1$s whose adjacency graph is precisely the McKay graph.

So Hilbert schemes show up in resolutions of Kleinian singularities! Therefore it should be no surprise that Nakajima quiver varieties will also show up.

Theorem 5.20: Let $G \subset \mathrm{SL}(2, \mathbb{C})$ be a finite subgroup, and Q the McKay graph. (The vertices of Q are indexed by the irreducible representations ρ_i of G.) Denote the vertex corresponding to the trivial G-representation by 0. Then fix

$$\mathbf{v} \coloneqq \delta,$$

i.e., $\mathbf{v}_i = \dim \rho_i$. On the other hand fix $\mathbf{w}$ to be 1 at vertex 0 (corresponding to the trivial representation) and 0 elsewhere. Then for $n = |G|$ and $\theta < 0$, we have:

$$\mathcal{M}_0(\mathbf{v}, \mathbf{w}) \simeq \mathbb{C}^2/G, \quad \mathcal{M}_\theta(\mathbf{v}, \mathbf{w}) \simeq \widetilde{\mathbb{C}^2/G},$$

the minimal resolution of $\mathbb{C}^2/G$ (for example, given by blowing up the origin enough times, or X^δ via the Hilbert scheme approach above). Furthermore, the canonical projective morphism

$$\pi : \mathcal{M}_\theta(\mathbf{v}, \mathbf{w}) \to \mathcal{M}_0(\mathbf{v}, \mathbf{w})$$

coincides with the minimal resolution of singularities (which also coincides with the Hilbert-Chow morphism by the previous theorem).

Remark 5.21: In fact you can take $\theta \in \mathbb{Z}^I$ to be $\mathbf{v}$-generic. Furthermore, for $\delta \neq \mathbf{v} \in \mathbb{Z}^I_{\geq 0}$, then $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w})$ (still $\theta < 0$) gets identified with other $X^{\mathbf{v}}$, defined in a similar manner to X^δ above. (We still keep $\mathbf{w}$ as before.)

For $k > 0$, it's even possible to choose θ so that

$$\mathcal{M}_\theta(k\delta, \mathbf{w}) \simeq \mathrm{Hilb}^k\left(\widetilde{\mathbb{C}^2/G}\right),$$

extending the previous result (which is $k = 1$).

Remark 5.22: One part of the McKay correspondence explains that the exceptional fiber in the minimal resolution of the Kleinian singularity is given by $\mathbb{P}^1$s, whose adjacency graph is the McKay graph (technically, that would be an extended Dynkin diagram, but we want to throw out the vertex corresponding to the trivial representation, and this recovers a Dynkin diagram). In fact we can more or less explicitly understand the exceptional fiber here by interpreting the minimal resolution $\widetilde{\mathbb{C}^2/G} \to \mathbb{C}^2/G$ as Nakajima quiver varieties, as discussed in the previous theorem. The irreducible components will end up being (special cases of) Hecke correspondences, see §5.6.2; they will end up all being $\mathbb{P}^1$s, and the irreducible components in the exceptional fiber. This is the same exceptional fiber studied in §5.3. Therefore the irreducible components of the exceptional fiber are in bijection with the vertices of the Dynkin diagram, i.e., nontrivial irreducible representations of G.

5.4.4 Torsion-free sheaves

Let X be an algebraic variety, and $\mathcal{F}$ a *coherent* sheaf. Recall that $\mathcal{F}$ is **torsion-free** if for every affine open $U \subset X$, then $\mathcal{F}(U)$ is torsion-free as an $\mathcal{O}_X(U)$-module. (This definition can be made for quasicoherent sheaves as well, but we only want to consider coherent sheaves for the purposes of this subsubsection.) A typical example is a vector bundle, i.e. a locally free sheaf.

Let us not consider torsion-free sheaves on $X = \mathbb{P}^2$. In fact they are *nearly* vector bundles:

Theorem 5.23: Let X be a nonsingular variety of dimension 2, and let $\mathcal{F}$ be a coherent torsion-free sheaf. Then there exists a dense affine open $U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to a vector bundle on U. We denote the **rank of** $\mathcal{F}$ to be the rank of its restriction to U, viewed as a vector bundle.

Let $\ell_\infty \coloneqq \{[0 : z_1 : z_2]\} \subset \mathbb{P}^2$ be the line at infinity, so that $\mathbb{P}^2 - \ell_\infty \simeq \mathbb{C}^2$.

Definition 5.24 (framing of torsion-free sheaf): Let $\mathcal{F}$ be a coherent torsion-free sheaf of rank r on $\mathbb{P}^2$. A **framing** of $\mathcal{F}$ is an isomorphism

$$\Psi : \mathcal{F}_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}.$$

Definition 5.25: The **moduli space of framed torsion sheaves on** $\mathbb{P}^2$ is defined by

$$\mathcal{M}^{\mathrm{fr}}(n, r) \coloneqq \{\text{isomorphism classes of pairs } (\mathcal{F}, \Psi)\},$$

where $\mathcal{F}$ is a torsion-free sheaf of rank r on $\mathbb{P}^2$ with framing Ψ, and $c_2(\mathcal{F}) = n$, where c_2 is the second Chern class (computed by using a locally free resolution of $\mathcal{F}$ viewed as a coherent sheaf).

As written, this is just a set, but it can naturally be given the structure of a scheme, and it is a fine moduli space of framed torsion-free sheaves.

Remark 5.26: The existence of framing implies that $c_1(\mathcal{F}) = 0$.

As you might expect given the subject of these notes, quiver varieties come up again.

Theorem 5.27: Let $n, r \geq 1$, and $\theta < 0$. Then

$$\mathcal{M}^{\mathrm{fr}}(n, r) \simeq \mathcal{M}_\theta(n, r),$$

the Nakajima quiver variety for the Jordan quiver.

Remark 5.28: For $r = 1$, note that

$$\mathcal{M}^{\mathrm{fr}}(n, 1) \simeq \operatorname{Hilb}^n \mathbb{C}^2 \simeq \mathcal{M}_\theta(n, 1),$$

recovering the previous identification.

Remark 5.29: Under the isomorphism $\mathcal{M}^{\mathrm{fr}}(n, r) \simeq \mathcal{M}_\theta(n, r)$, the subset $\mathcal{M}_(n, r)^{\mathrm{reg}} \subset \mathcal{M}_(n, r)$ gets identified with the moduli space of framed *vector bundles* of rank r with $c_2(\mathcal{F}) = n$ (which is of course a subset of the moduli space of framed torsion-free sheaves of rank r with $c_2(\mathcal{F}) = n$).

This also holds for $r = 1$, but both sides are empty; there are no nontrivial framed line bundles on $\mathbb{P}^2$, and there are no regular points in $\mathcal{M}_\theta(n, 1)$.

Remark 5.30: One can more explicitly decompose $\mathcal{M}_0(n, r)$, and describe the canonical morphism $\pi : \mathcal{M}_\theta(n, r) \to \mathcal{M}_0(n, r)$ rather explicitly in terms of the sheaf that each point corresponds to. See [KJ16, Theorem 11.21].

Remark 5.31: We'll conclude this subsection with one more remark: the Nakajima quiver varieties $\mathcal{M}_0(n,r)$ and $\mathcal{M}_\theta(n,r)$ are also closely related to moduli spaces of framed (rank r) n-instantons on $\mathbb{R}^4$ (also known as anti-self-dual connections on $\mathbb{R}^4$). This description was first introduced by Atiyah-Drinfeld-Hitch-Manin [AHDM94], and is usually called the ADHM construction.

5.5 A Steinberg-type variety

In this subsection we only consider the case $\lambda = 0$; as a result let us save space and denote

$$\mathcal{M}_\theta(\mathbf{v},\mathbf{w}) \coloneqq \mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w}).$$

We also require Q to have no edge-loops (i.e. loops of length 1).

Definition 5.32: Given $\theta \in \mathbb{Z}^I$ and $\mathbf{v},\mathbf{v}' \in \mathbb{Z}^I_{\geq 0}$ two dimension vectors, we define the **associated Steinberg variety**

$$Z_\theta(\mathbf{v},\mathbf{v}',\mathbf{w}) \coloneqq \mathcal{M}_\theta(\mathbf{v},\mathbf{w}) \times_{\mathcal{M}_0(\mathbf{v}+\mathbf{v}',\mathbf{w})} \mathcal{M}_\theta(\mathbf{v}',\mathbf{w}) \subset \mathcal{M}_\theta(\mathbf{v},\mathbf{w}) \times \mathcal{M}_\theta(\mathbf{v}',\mathbf{w}).$$

Let me explain how this works. First we have a decomposition

$$\mathbb{C}^{(\mathbf{v}+\mathbf{v}')_i} \simeq \mathbb{C}^{\mathbf{v}_i} \oplus \mathbb{C}^{\mathbf{v}'_i}.$$

Therefore we have a vector space embedding on the space of framed representations

$$R(Q^\#,\mathbf{v},\mathbf{w}) \hookrightarrow R(Q^\#,\mathbf{v}+\mathbf{v}',\mathbf{w}), \quad (z,j,i) \mapsto (z,j,i) \oplus \mathbf{0}'.$$

This induces a morphism of the GIT quotients

$$\mathcal{M}_0(\mathbf{v},\mathbf{w}) \to \mathcal{M}_0(\mathbf{v}+\mathbf{v}',\mathbf{w}).$$

Remark 5.33: This does **not** give a natural morphism

$$\mathcal{M}_\theta(\mathbf{v},\mathbf{w}) \to \mathcal{M}_\theta(\mathbf{v}+\mathbf{v}',\mathbf{w}),$$

because the stability conditions are not compatible (in general).

Lemma 5.34: In fact the natural morphism

$$\mathcal{M}_0(\mathbf{v},\mathbf{w}) \to \mathcal{M}_0(\mathbf{v}+\mathbf{v}',\mathbf{w})$$

is a closed embedding.

Now we can realize the map

$$\mathcal{M}_\theta(\mathbf{v},\mathbf{w}) \to \mathcal{M}_0(\mathbf{v}+\mathbf{v}',\mathbf{w})$$

as the composition of the natural maps

$$\mathcal{M}_\theta(\mathbf{v},\mathbf{w}) \to \mathcal{M}_0(\mathbf{v},\mathbf{w}) \hookrightarrow \mathcal{M}_0(\mathbf{v}+\mathbf{v}',\mathbf{w}).$$

As a result we have a natural projective morphism

$$\pi_Z : Z_\theta(\mathbf{v},\mathbf{v}',\mathbf{w}) \to \mathcal{M}_0(\mathbf{v}+\mathbf{v}',\mathbf{w}).$$

Remark 5.35: The Steinberg variety is typically quite singular and has many irreducible components. However, the dimension of any irreducible component of $Z_\theta(\mathbf{v},\mathbf{v}',\mathbf{w})$ is bounded above by

$$\frac{\dim \mathcal{M}_\theta(\mathbf{v},\mathbf{w}) \times \mathcal{M}_\theta(\mathbf{v}',\mathbf{w})}{2}.$$

Nakajima proved that in the case of Q being either a finite Dynkin or extended Dynkin quiver, then in fact equality is reached, i.e. each irreducible component of $Z_\theta(\mathbf{v},\mathbf{v}',\mathbf{w})$ has dimension exactly equal to $\frac{\dim \mathcal{M}_\theta(\mathbf{v},\mathbf{w}) \times \mathcal{M}_\theta(\mathbf{v}',\mathbf{w})}{2}$.

5.6 Geometric construction of $\widetilde{\mathcal{U}}(\mathfrak{g}_Q)$

We again need Q to have no edge loops.

5.6.1 $\widetilde{\mathcal{U}}(\mathfrak{g}_Q)$

This will be a very brief review, more to set notation than to explain much. For more details see [KJ16, Appendix A].

Associated to a finite **graph** Q (with no edge loops) is the **generalized Cartan matrix** C_Q. This generalized Cartan matrix satisfies certain important properties, such as being symmetric. It is indecomposable iff Q is connected, so let us assume that. Now from this generalized Cartan matrix we can define simple roots, the root lattice, the weight lattice, the Weyl group, etc. What is important is we can construct a complex Lie algebra called the **Kac-Moody Lie algebra** $\mathfrak{g}(C)$ associated with C, and it has a Cartan decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

along with root decompositions. For example, when Q is a Dynkin graph we get the classical finite-dimensional simple Lie algebras; when Q is an extended Dynkin graph then we get the affine Kac-Moody Lie algebras.

To any Lie algebra $\mathfrak{g}$ we can construct its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Choosing a polarization gives us a polarization

$$\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{n}_-) \otimes \mathrm{Sym}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{n}_+).$$

Notice that $\mathrm{Sym}(\mathfrak{h}) \simeq \mathbb{C}[\mathbb{Z}_{\geq 0}\Phi]$.

Definition 5.36: We define the **modified universal enveloping algebra** $\widetilde{\mathcal{U}}(\mathfrak{g})$ to be

$$\widetilde{\mathcal{U}}(\mathfrak{g}) \coloneqq \mathcal{U}(\mathfrak{n}_-) \otimes \mathbb{C}[\Lambda] \otimes \mathcal{U}(\mathfrak{n}_+),$$

where Λ is the weight lattice.

In other words, we replace the root monoid with the weight lattice. However, a **warning:** the $\mathbb{C}[\Lambda]$ is only suggestive as a vector space; it is not how the multiplication works!

The multiplication is given as follows. Let us denote a_λ by the basis element in $\mathbb{C}[\Lambda]$ corresponding to λ. Then multiplication in $\widetilde{\mathcal{U}}(\mathfrak{g})$ is defined by

$$\begin{aligned} a_\lambda a_\mu &= \delta_{\lambda=\mu} a_\lambda, \\ e_i a_\lambda &= a_{\lambda+\alpha_i} e_i, \\ f_i a_\lambda &= a_{\lambda-\alpha_i} f_i, \\ (e_i f_k - f_k e_i) a_\lambda &= \delta_{i=k} \langle \lambda, \alpha_i^\vee \rangle a_\lambda. \end{aligned}$$

5.6.2 Hecke correspondences

We'll take $\lambda = 0$ again, and now we require $\theta > 0$ to be positive, i.e., $\theta_i > 0$ for all $i \in I$. Take $\mathbf{v} \in \mathbb{Z}^I_{\geq 0}$. Choose a vertex $i \in I$ and let $\alpha_i \in \mathbb{Z}^I_{\geq 0}$ be the corresponding simple root.

Definition 5.37 (Hecke correspondence): Define

$$B_i(\mathbf{v},\mathbf{w}) \coloneqq \{(m,n,\varphi) \mid m \in \mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w}), \quad n \in \mathcal{M}_{\theta,0}(\mathbf{v}+\alpha_i,\mathbf{w}), \quad \varphi : V(m) \longrightarrow V(m')\}$$

so that φ is a morphism of framed representations. It easily follows from stability of m, n that if φ exists, it is unique and injective. This implies that

$$B_i(\mathbf{v},\mathbf{w}) \subset \mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w}) \times \mathcal{M}_{\theta,0}(\mathbf{v}+\alpha_i,\mathbf{w}).$$

We call the varieties $B_i(\mathbf{v},\mathbf{w})$ **Hecke correspondences**.

Remark 5.38: We can also describe $B_k(\mathbf{v},\mathbf{w})$ as the set of isomorphism classes of pairs (V, V'), where V' is a stable $\mathbf{w}$-framed representation of dimension $\mathbf{v}+\alpha_k$, and $V \subset V'$ is a $Q^\#$-subrepresentation of dimension $\mathbf{v}$ such that the images of the morphisms $i_t : W_t \longrightarrow V'_t$ actually lie in V_t. In this case, the quotient $V'/V \simeq S(k)$ is the one-dimensional simple representation of $Q^\#$ at vertex k.

The important part is that

Definition 5.39: $B_i(\mathbf{v},\mathbf{w})$ is a smooth Lagrangian subvariety in $\mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w}) \times \mathcal{M}_{\theta,0}(\mathbf{v}+\alpha_i,\mathbf{w})$. In particular its dimension is given by

$$\dim B_i(\mathbf{v},\mathbf{w}) = \frac{\dim \mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w}) \times \mathcal{M}_{\theta,0}(\mathbf{v}+\alpha_i,\mathbf{w})}{2}$$

and hence it is an irreducible component of $Z(\mathbf{v},\mathbf{v}+\alpha_i,\mathbf{w})$.

This will end up being very important, because we will construct the algebra via Borel-Moore homology, which (in the top degree) has a basis given by irreducible components.

5.6.3 The algebra

Once again set $\lambda = 0$ and $\theta > 0$ to be positive. Choose $\mathbf{w} \in \mathbb{Z}^I_{\geq 0}$. Our strategy is to use Borel-Moore homology of Steinberg varieties to obtain an algebra, much akin to the way the Springer correspondence works. (However, I won't explain Borel-Moore homology here.)

We will consider the space

$$H^{BM}_\bullet(\mathbf{w}) \coloneqq \bigoplus_{\mathbf{v},\mathbf{v}' \in \mathbb{Z}^I_{\geq 0}} H^{BM}_\bullet(Z(\mathbf{v},\mathbf{v}',\mathbf{w})).$$

It is clear from the construction of the Steinberg varieties that the convolution product gives $H^{BM}_\bullet(\mathbf{w})$ the structure of an associative algebra, since

$$Z(\mathbf{v},\mathbf{v}',\mathbf{w}) \circ Z(\mathbf{v}',\mathbf{v}'',\mathbf{w}) = Z(\mathbf{v},\mathbf{v}'',\mathbf{w}),$$

along with some other properties for convolution to hold (such as properness). Now as is frequently the case with Borel-Moore homology we're really interested in the *top* spaces. Since

$$\dim_{\mathbb{C}} Z(\mathbf{v},\mathbf{v}',\mathbf{w}) = \frac{\dim \mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w}) \times \mathcal{M}_{\theta,0}(\mathbf{v}',\mathbf{w})}{2},$$

and Borel-Moore homology works with *real* dimension, we set

$$H^{BM}_{\text{top}}(\mathbf{w}) \coloneqq \bigoplus_{\mathbf{v},\mathbf{v}' \in \mathbb{Z}^I_{\geq 0}} H_{\dim \mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w}) + \dim \mathcal{M}_{\theta,0}(\mathbf{v}',\mathbf{w})}(Z(\mathbf{v},\mathbf{v}',\mathbf{w})).$$

This is a subalgebra in $H^{BM}_\bullet(\mathbf{w})$ with respect to the convolution product. Importantly, however, $H^{BM}_{\text{top}}(\mathbf{w})$ has a natural basis given by the fundamental classes of the irreducible components of $Z(\mathbf{v},\mathbf{v}',\mathbf{w})$. Some distinguished elements we can write:

- $[\Delta_{\mathbf{v}}] \in H_{\text{top}}^{BM}(Z(\mathbf{v},\mathbf{v},\mathbf{w}))$. This is the diagonal inside $\mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w}) \times \mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w})$.
- $[B_i(\mathbf{v},\mathbf{w})] \in H_{\text{top}}^{BM}(Z(\mathbf{v},\mathbf{v}+\alpha_i,\mathbf{w}))$. This is the Hecke correspondence.
- $[B_i^{\intercal}(\mathbf{v},\mathbf{w})] \in H_{\text{top}}^{BM}(Z(\mathbf{v},\mathbf{v}+\alpha_i,\mathbf{w}))$. This is the Hecke correspondence after applying the transposition swapping the two factors of $\mathcal{M}$.

We can now geometrically "realize" parts of $\widetilde{\mathcal{U}}(\mathfrak{g}_Q)$. Let ω_i denote the fundamental weights, and let $\omega(\mathbf{w}) \coloneqq \sum_{i\in I} \mathbf{w}_i\omega_i$. Similarly let α_i denote the simple roots, and $\alpha(\mathbf{v}) \coloneqq \sum_{i\in I} \mathbf{v}_i\alpha_i$.

Theorem 5.40: Let Q be a graph without edge loops, and let $\mathbf{w} \in \mathbb{Z}^I_{|ge0}$. Then there exists a unique algebra homomorphism

$$\Phi_{\mathbf{w}} : \widetilde{\mathcal{U}}(\mathfrak{g}_Q) \to H_{\text{top}}^{BM}(\mathbf{w})$$

such that:

- if $\omega(\mathbf{w}) - \lambda = \alpha(\mathbf{v})$ for some $\mathbf{v} \in \mathbb{Z}^I_{\geq 0}$, then:
  - $\Phi_{\mathbf{w}}(a_\lambda) = [\Delta_{\mathbf{v}}]$,
  - $\Phi(e_i a_\lambda) = [B_i(\mathbf{v}-\alpha_i,\mathbf{w})]$,
  - $\Phi(f_i a_\lambda) = (-1)^{1+\langle\lambda,\alpha_i^\vee\rangle}[B_i^{\intercal}(\mathbf{v},\mathbf{w})]$.
- if $\omega(\mathbf{w}) - \lambda$ is not equal to $\alpha(\mathbf{v})$ for some $\mathbf{v} \in \mathbb{Z}_{\geq 0}$, then $\Phi_{\mathbf{w}}(a_\lambda) = 0$.

This can be proved by an explicit computation of convolution inside $H_{\text{top}}^{BM}(\mathbf{w})$.

5.6.4 Constructing representations

A very similar method gives rise to geometric constructions of $\widetilde{\mathcal{U}}(\mathfrak{g}_Q)$-modules. The baby example is as follows. To each map $\pi : \mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w}) \to \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$, let $\mathcal{L}_\theta(\mathbf{v},\mathbf{w}) \coloneqq \pi^{-1}(0)$ denote the fiber over 0, the exceptional fiber considered in §5.3. Then:

Theorem 5.41: Let

$$L_{\mathbf{w}} \coloneqq \bigoplus_{\mathbf{v}\in\mathbb{Z}^I_{\geq 0}} H_{\text{top}}^{BM}(\mathcal{L}_\theta(\mathbf{v},\mathbf{w})).$$

This is naturally a $H_\bullet^{BM}(\mathbf{w})$-module. Via the map (described in Theorem 5.40)

$$\Phi_{\mathbf{w}} : \widetilde{\mathcal{U}}(\mathfrak{g}_Q) \to H_{\text{top}}^{BM}(\mathbf{w}),$$

$L_{\mathbf{w}}$ is a simple integrable $\widetilde{\mathcal{U}}(\mathfrak{g}_Q)$-module, hence a simple integrable $\mathfrak{g}_Q$-module, of highest weight $\omega(\mathbf{w})$.

This construction generalizes fairly easily to give us basically every other irreducible representation.

Theorem 5.42: Fix $\mathbf{u}, \mathbf{w} \in \mathbb{Z}^I_{\geq 0}$. Pick a point $x \in \mathcal{M}_{0,0}(\mathbf{u}, \mathbf{w})^{\text{reg}}$ (so we want it to be nonempty). For every $\mathbf{v} \in \mathbb{Z}^I_{\geq 0}$ such that $\mathbf{v} - \mathbf{u} \in \mathbb{Z}^I_{\geq 0}$, we have a closed embedding (see §5.5)

$$\mathcal{M}_{0,0}(\mathbf{u}, \mathbf{w}) \hookrightarrow \mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w}),$$

so we may view point x as a point in $\mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w})$. We also have the canonical projective morphism

$$\pi : \mathcal{M}_{\theta,0}(\mathbf{v}, \mathbf{w}) \to \mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w}).$$

Then:

(a) The fiber $\mathcal{M}_{\theta,0}(\mathbf{v}, \mathbf{w})_x \coloneqq \pi^{-1}(x)$ is equi-dimensional.
(b) The weight $\lambda \coloneqq \omega(\mathbf{w}) - \alpha(\mathbf{u})$ is dominant.
(c) The vector space

$$L_{\mathbf{u},\mathbf{w}} \coloneqq \bigoplus_{\mathbf{v}-\mathbf{u}\in\mathbb{Z}^I_{\geq 0}} H^{BM}_{\text{top}}(\mathcal{M}_{\theta,0}(\mathbf{v}, \mathbf{w})_x)$$

is a module over $\widetilde{\mathcal{U}}(\mathfrak{g}_Q)$ via the map $\Phi_{\mathbf{w}}$. In fact it is the simple integrable module of highest weight $\lambda = \omega(\mathbf{w}) - \alpha(\mathbf{u})$.
(d) Additionally, each individual component $H^{BM}_{\text{top}}(\mathcal{M}_{\theta,0}(\mathbf{v}, \mathbf{w})_x)$ inside the irreducible representation $L_{\mathbf{u},\mathbf{w}}$ is identified with the weight space of $\mu \coloneqq \omega(\mathbf{w}) - \alpha(\mathbf{v})$.

Remark 5.43: This strategy, to 1) realize some Borel-Moore algebra as an image of $\widetilde{\mathcal{U}}(\mathfrak{g}_Q)$, and 2) obtain all representations of $\widetilde{\mathcal{U}}(\mathfrak{g}_Q)$, is very similar to the approach taken in [CG97, §4] to geometrically construct all representations of $\mathfrak{sl}_n$. In fact this is not a coincidence: as we saw earlier in Example 4.28, for $\overrightarrow{Q}$ a type A quiver, $\theta > 0$, and for certain choices of $\mathbf{v}, \mathbf{w}$, then $R_\theta(\mathbf{v}, \mathbf{w})$ is a partial flag variety $\mathcal{F}\ell(\mathbf{v}, r)$. In fact it turns out that the natural projection

$$\pi : \mathcal{M}_{\theta,0}(\mathbf{v}, \mathbf{w}) \to \mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w})$$

is *exactly* the moment map

$$\mu : T^*\mathcal{F}\ell(\mathbf{v}, r) \to \mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w}) \subset \mathfrak{gl}_r,$$

although $\mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w})$ can be somewhat difficult to describe. (If certain conditions on $\mathbf{v}$ are satisfied then $T^*\mathcal{F}\ell(\mathbf{v}, r) \to \mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w})$ is surjective and $\mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w})$ is identified with with the closure of a nilpotent orbit, hence $\pi \leftrightarrow \mu$ becomes a symplectic resolution of singularities.) Then Nakajima's results on constructing the representations of $\widetilde{\mathcal{U}}(\mathfrak{g}_Q)$ can be specialized, in the case of Q a type A Dynkin graph, to geometrically constructing the representations of $\mathfrak{sl}_n$ via the Springer fibers of the partial flag varieties.

Remark 5.44: Similar approaches can be done for other important algebras. For example the Heisenberg algebra can be geometrically realized by Hilbert schemes (of smooth quasiprojective surfaces). Another example is the quantized universal enveloping algebra of the affinization of $\mathfrak{g}_Q$; this is realized by replacing cohomology by equivariant K-theory.

Remark 5.45: Another important application of the geometric construction in Theorem 5.42 is that it gives us a distinguished basis in $L(\omega(\mathbf{w}))$. For every $\mathbf{v}$, we have a natural basis in $H^{BM}_{\text{top}}(\mathcal{L}_\theta(\mathbf{v},\mathbf{w}))$ given by the fundamental classes of the irreducible components of $\mathcal{L}_\theta$. (Recall from §5.3 that these are closures of Bialynicki-Birula pieces.) This basis is called the **semicanonical basis**, and was shown by Saito to have a special combinatorial structure called a **crystal basis**.

In fact this semicanonical basis has a bijection to a subset of the semicanonical basis (constructed by Lusztig) of $\mathcal{U}(\mathfrak{n}_-)$. Let me briefly sketch out the main ideas. In Remark 5.11 we make mention of $\Lambda(\mathbf{v})$, a variety of nilpotent representations of $Q^\#$. From this variety one can construct a **composition algebra**, which in the case of Q being a grpah without edge loops, is isomorphic to $\mathcal{U}(\mathfrak{n}_-)$, where $\mathfrak{n}_-$ is the negative part of the Kac-Moody Lie algebra $\mathfrak{g}_Q$. This has a **semicanonical basis** which is constructed geometrically by the irreducible components of $\Lambda(\mathbf{v})$. In Remark 5.11 we noted that there is a natural injective map of irreducible components from $\mathcal{L}_\theta(\mathbf{v},\mathbf{w})$ to $\Lambda(\mathbf{v})$. This gives us an identification of the semicanonical basis of $L(\omega(\mathbf{w}))$ (as the union over the fundamental classes of the irreducible components of all $\mathcal{L}_\theta(\mathbf{v},\mathbf{w})$) with a *subset* of Lusztig's semicanonical basis of $\mathcal{U}(\mathfrak{n}_-)$, given by irreducible components of $\Lambda(\mathbf{v})$.

Note that all Verma modules $M(\lambda)$ are a free rank one $\mathcal{U}(\mathfrak{n}_-)$-module; then the irreducible $L(\lambda) \coloneqq M(\lambda)/J$, for the maximal proper submodule J. Then the semicanonical basis for $\mathcal{U}(\mathfrak{n}_-)$ must necessarily give a basis for $M(\lambda)$. The amazing part is that this basis in fact is compatible with the submodule J, and the remainder of Lusztig's semicanonical basis exactly recovers the semicanonical basis of $L(\lambda)$ constructed geometrically (after identifying as a subset of Lusztig's semicanonical basis).

In other words, identifying the geometric semicanonical basis here with a subset of Lusztig's semicanonical basis, then that subset descends to give the geometric basis of $L(\lambda)$, while the remainder of Lusztig's semicanonical basis in fact formed a basis of J.

In fact the same property also holds for Lusztig's canonical basis, but this is not the same as Lusztig's semicanonical basis.

In particular, find a criterion for when $\mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w})$ is nonempty.

Theorem 5.46: Let $\mathbf{v},\mathbf{w} \in \mathbb{Z}^I_{\geq 0}$, and let $\theta > 0$. Then:

(a) The following conditions are equivalent:
- $\mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w}) \neq \emptyset$;
- $\mathcal{L}_\theta(\mathbf{v},\mathbf{w}) \neq \emptyset$, where $\mathcal{L}_\theta(\mathbf{v},\mathbf{w}) \coloneqq \pi^{-1}(0) \subset \mathcal{M}_{\theta,0}(\mathbf{v},\mathbf{w})$;
- $\lambda \coloneqq \omega(\mathbf{w}) - \alpha(\mathbf{v})$ is a (nonempty) weight space of $L(\omega(\mathbf{w}))$, the irreducible representation of highest weight $\omega(\mathbf{w})$.

(b) If $\mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})^{\text{reg}} \neq \emptyset$, then $\lambda \coloneqq \omega(\mathbf{w}) - \alpha(\mathbf{v})$ is a dominant weight, and is a weight of $L(\omega(\mathbf{w}))$.

(c) We have a partial converse to part (b) as follows. Assume additionally that for any positive imaginary root α, that
$$(\alpha, \omega(\mathbf{w})) \geq 2.$$
Then the converse to part (b) holds: if $\lambda \coloneqq \omega(\mathbf{w}) - \alpha(\mathbf{v})$ is a dominant weight and is a weight of $L(\omega(\mathbf{w}))$, then $\mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})^{\text{reg}} \neq \emptyset$.

Proof sketch. The first two parts are essentially applying the previous two theorems; Theorem 5.41 gives us the representation, while Theorem 5.42 identifies $H^{BM}_{\text{top}}(\mathcal{L}_\theta(\mathbf{v},\mathbf{w}))$ as the weight space of weight $\omega(\mathbf{w}) - \alpha(\mathbf{v})$. The third part is much harder and we won't discuss it. ◻

Index

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